BLOB ALGEBRA APPROACH TO MODULAR REPRESENTATION THEORY

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Abstract. Two decades ago P. Martin and D. Woodcock made a surprising (and prophetic) link between statistical mechanics and representation theory. They observed that the decomposition numbers of the blob algebra (that appeared in the context of transfer matrix algebras) are Kazhdan-Lusztig polynomials in type $\tilde{A}_1$. In this paper we take that observation far beyond its original scope. We conjecture (and prove for $\tilde{A}_1$, $p \neq 2$) an algorithm to produce the p-Kazhdan-Lusztig polynomials in type $\tilde{A}_n$, using "generalized blob algebras", that are quotients of certain KLR algebras. It is of fundamental importance for modular representation theory to understand these polynomials, as they control the modular representation theory of the symmetric group and of the special linear group, by recent work of G. Williamson and S. Riche. We believe that our conjecture is the shadow of an equivalence between the Hecke category for $\tilde{A}_n$ and a certain "Blob category", that we introduce.

1. Introduction

1.1. **A new paradigm.** Kazhdan-Lusztig polynomials have been at the heart of representation theory since their discovery in 1979. They have answered (often via geometric methods) an enormous number of deep questions concerning the characteristic zero and characteristic $p \gg 0$ representation theory of Weyl groups, Lie algebras, quantum groups, and reductive algebraic groups. It was also widely believed (see, for example, Lusztig and James's conjectures) that these polynomials control the modular (i.e., characteristic p) representation theory of these structures if p is not too small[1].

A new paradigm has emerged in the last few years by the work of G. Williamson and his collaborators (see [RW15, JW17, Wil17b, Wil17a, AMRW17]). Now we know that p-Kazhdan Lusztig polynomials are the correct objects of study in modular representation theory of Lie-type objects. They provide solutions to old questions such as the modular representation theory of $SL_n(F_q)$ (with q a power of p) or the modular representation theory of the symmetric group S_n. There is even a geometric meaning to them! Just as Kazhdan-Lusztig polynomials have an interpretation in terms of the stalks of intersection cohomology complexes on flag varieties, p-Kazhdan-Lusztig polynomials have one in terms of the stalks of parity sheaves with coefficients in a field of characteristic p.

But there is still a big open problem in the theory. Kazhdan-Lusztig polynomials are obtained by calculating in an algebra (the Hecke algebra) but p-Kazhdan-Lusztig polynomials are obtained by calculating in a category (the Hecke category). They don't seem to be a satisfactory answer to the question. We need to understand better the p-Kazhdan-Lusztig polynomials in order to have a "real" answer[2]. A step in this direction is given by the exciting conjecture (and theorem for very small p) by G. Lusztig and G. Williamson [LW] where they predict that the characters of certain tilting modules for SL_3 (i.e., certain p-Kazhdan-Lusztig

[1] To be *not too small* depends on the case and on the author. For example, in Lusztig conjecture for $SL_n(\overline{\mathbb{F}}_p)$, in Lusztig's original formulation the bound was $p > 2n-3$ and the bound in Kato's version was $p > n$. In James's conjecture for S_n the bound was $p > \sqrt{n}$.

[2] It is a matter of discussion what should one expect as a "real" answer. Kazhdan-Lusztig polynomials are already quite difficult objects, and their p-version is harder. The authors hope for some kind of answer in the vein of Kazhdan-Lusztig polynomials, i.e., some recursive algorithm in some algebra or at least something more enlightening than calculating ranks of intersection forms.

polynomials for $\tilde{A}_2$) are governed by a discrete dynamical system, that looks like billiards bouncing in equilateral triangles. But this conjecture also shows how incredibly complex is this quest and how far are we from a full understanding of the p-canonical basis.

We see this paper as a link between physics and modular representation theory and we hope that it will help to raise intuitions from physics towards the (nowadays) obscure land of modular representation theory. The authors are working to find a statistical mechanical model for the generalized blob algebra in characteristic p.

1.2. **Elias-Williamson's Hecke category.** The term p-canonical basis was coined by I. Grojnowski [Gro] in 1999. Its modern use (i.e., where one can actually calculate) comes from G. Williamson [Wil12], even though both definitions are known to coincide. For this definition, we need first to explain a categorification of the Hecke algebra.

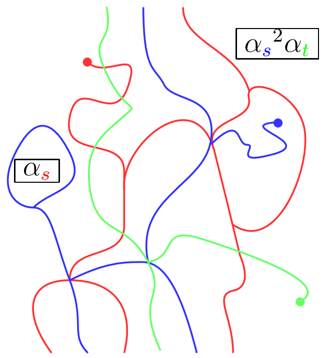
In [EW16] it was defined, for any Coxeter system (W, S) the *diagrammatic Hecke category* $\mathcal{H}$ (or $\mathcal{H}(W)$ when the group is not clear from the context). It is also called in the literature the Elias-Williamson's *diagrammatic Soergel category*[3]. (Versions of this category for particular groups were found before by [Lib10], [EK10], [Eli16]).

The definition goes roughly like this. Firstly we need to define the *Bott-Samelson diagrammatic category* $\mathcal{H}_{\mathrm{BS}}$ (for details see sections 2.3 and 2.4). Assign a different color to each element of S. Objects in the category $\mathcal{H}_{\mathrm{BS}}$ correspond to sequences of colored dots. For $\underline{w} = sr\cdots t$ the corresponding object is

$$BS(\underline{w}) := \bullet \bullet \cdots \bullet.$$

Let k be a field (we could even take k to be the ring of integers in this definition and then extend scalars) and let $R := k[\alpha_s]_{s\in S}$ be the polynomial ring in variables α_s parametrized by the set of simple reflections S. The ring R is endowed with an action of W and with an operator $\partial_s : R \to R$ that we do not make explicit in this introduction.

The morphisms in $\mathcal{H}_{\mathrm{BS}}$ are (modulo some local relations) linear combinations over R of isotopy classes of some decorated planar graphs embedded in the strip $\mathbb{R} \times [0, 1]$. For example, a morphism between $\bullet \bullet \bullet \bullet \bullet \bullet$ (bottom) and $\bullet \bullet \bullet$ (top) looks like



(1.1)

[3]There are (at least) two other versions of this category. The first one is an algebraic version called the category of *Soergel bimodules*. It is an additive monoidal full subcategory of the category of $\mathbb{Z}$-graded R-bimodules, where R is some ring of polynomials acted upon by W.

The other version is of geometric origin. Let (W, S) be the Weyl group with simple reflections corresponding to a pair (G, B), where G is a complex reductive group and $B \subset G$ is a Borel subgroup. This version of the Hecke category is the additive, monoidal (under convolution) category of B-biequivariant semi-simple complexes on G.

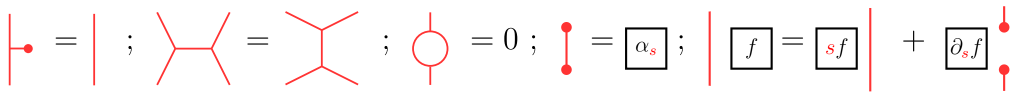
By explaining things like this, we are going backwards in time. The geometric category was the first one that emerged. Soergel bimodules is a simplified version with the advantages that 1) It is defined for all Coxeter groups and 2) It is way easier to compute with. The diagrammatic Hecke category is a simplified version of Soergel bimodules with the advantages that 1) It is well-behaved even over fields of positive characteristic and 2) It is defined over the integers, it is only after an extension of scalars that it is equivalent to Soergel bimodules.

The edges of these graphs are colored as the elements of S and they may end in a dot with the same color of the boundary of the strip. The connected components of the complement of the diagram can be decorated by elements of R.
The generating morphisms, i.e., the kinds of vertices allowed in the graphs, are:

Dot ; Trivalent Vertex ; $2m_{s,t}$-valent vertex (here $m_{s,t} = 3$) . (1.2)

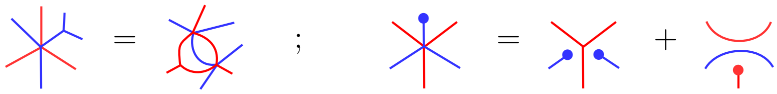
Finally, to obtain the morphisms in $\mathcal{H}_{\mathrm{BS}}$ one has to quotient the set of graphs obtained in this manner by the following local relations:

- One color relations:



(1.3)

- Two color relations (here we only illustrate the case $m_{s,t} = 3$):



(1.4)

- Three color relations (not described in this introduction).

To finish the definition of $\mathcal{H}_{\mathrm{BS}}$, one needs to enrich the morphism spaces with a grading. They have a unique grading if we prescribe that the dot, the trivalent and the $2m_{st}$-valent vertices have degrees $1, -1$ and 0 respectively.

Definition 1.1. *The* diagrammatic Hecke category $\mathcal{H}$ *is the Karoubian envelope of* $\mathcal{H}_{\mathrm{BS}}$ *(one formally adds direct summands of the objects, i.e. the objects in* $\mathcal{H}$ *are pairs* (M, e)*, with* M *an object of* $\mathcal{H}_{\mathrm{BS}}$ *and* e *an idempotent in* $\mathrm{End}(M)$*.)*

Let $(-)^a : \mathcal{H} \to \mathcal{H}$ be the endofunctor that flips upside down a diagram. For any element $x \in W$, fix some (arbitrary) reduced expression $\underline{x}$ of x. Given any reduced expression $\underline{w}$ of $w \in W$ and some $x \leq w$ the set of *light leaves* $\mathbb{L}_{\underline{w}}(x)$ (for details see section 2.7) is a combinatorially defined, finite subset of graded morphisms in $\mathrm{Hom}_{\mathcal{H}}(BS(\underline{w}), BS(\underline{x}))$. One can prove that, if $\underline{u}$ is a reduced expression of $u \in W$, then the set

$$\{ \mathbb{L}_{\underline{u}}(x)^a \circ \mathbb{L}_{\underline{w}}(x) \mid x \in W \} \tag{1.5}$$

is an R-basis of the set $\mathrm{Hom}_{\mathcal{H}}(BS(\underline{w}), BS(\underline{u}))$. It is called the *double leaves basis.*

1.3. **The p-canonical basis.** It is a theorem by Elias and Williamson [EW16] (following a similar theorem by Soergel in the context of Soergel bimodules [Soe92]) that the indecomposable objects B_w of $\mathcal{H}$ are parametrized (modulo grading shift) by the elements $w \in W$. These indecomposable objects are idempotents in $\mathrm{End}_{\mathcal{H}_{BS}} BS(\underline{w})$ for any reduced expression $\underline{w}$ of w, so they are some (very difficult to calculate in practice) linear combination of colored graphs. Moreover, B_w can be defined as the unique indecomposable object appearing in $BS(\underline{w})$ that has not appeared in any decomposition of $BS(\underline{x})$ for $x < w$.
We said before that the diagrammatic Hecke category behaves well even in positive characteristic. By that we meant that there is a canonical "character" isomorphism[4] of $\mathbb{Z}[v, v^{-1}]-$algebras

[4]This character is also an isomorphism if one replaces $\mathcal{H}$ by some categories of Soergel bimodules. It is the case for any Coxeter system using the geometric representation in characteristic zero, as proved by Soergel [Soe92] and the first author [Lib08a]. It is also the case for Soergel bimodules associated to Weyl groups, using the Cartan matrix representation in positive characteristic $\neq 2, 3$, by [Lib15]. It is still unknown for

$\mathrm{ch} : [\mathcal{H}] \to H$ between the split Grothendieck group of $\mathcal{H}$ and the Hecke algebra H of (W, S) [EW16]. This is true for any characteristic of the field k. Hence, $\mathcal{H}$ provides a categorification of the Hecke algebra.

Definition 1.2. *If W is a crystallographic Coxeter group and k has characteristic p, the* p-canonical basis *is the subset $\{\mathrm{ch}(B_x)\}_{x\in W}$ of the Hecke algebra. The coefficients $h^p_{x,y}$ of the base change matrix between the p-canonical basis and the standard basis*

$$\mathrm{ch}(B_y) = \sum_x h^p_{x,y} H_x$$

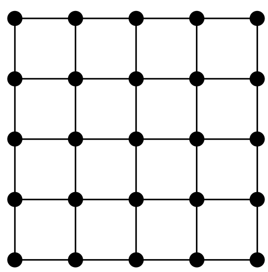
are the p-Kazhdan-Lusztig polynomials.

Implicit in this definition is the theorem ([JW17]) that if W is a crystallographic group, the characters of the indecomposable objects B_x depend only on the characteristic of the field k.

1.4. **Some statistical mechanics.** The goal of classical statistical mechanics is to consider a model and by studying its behavior on some small scale (usually atomic) obtain some large-scale result (usually macroscopic). In this field of physics, there is a model, the Potts model, that in spite of not corresponding to anything found in nature, it is extremely interesting to study[5]. This model consists roughly of a "geometry" (where do the atoms sit) a "configuration" (an angular momentum for each atom) and a "Hamiltonian" (how much energy has a given configuration).

The Potts model is interesting mainly for two reasons. Firstly, because it gives accurately properties close to phase transitions (examples of phase transitions are boiling or melting water after a change of temperature, or, cooling enough some iron, it transforms itself into a permanent magnet, thus having *spontaneous magnetization* below the Curie temperature). These properties are (surprisingly) quite insensitive to the Hamiltonian, so even though the Hamiltonian of this model does not appear in nature, by calculating this model one can predict phase transitions properties that do appear in nature. Secondly, this model is interesting because it is extremely rich mathematically and in some particular cases (very few of them) one can find an analytical expression of the partition function, which is an extraordinary achievement in physics.

Let us be (a bit) more precise. In the Potts model, you consider a graph, usually a subgraph of a Euclidean lattice. Let us consider for the rest of this section the example of the graph in (1.6) with vertices a set I of atoms and edges between *neighbors* of I. Each vertex i has some number of states s_i (a positive integer smaller or equal than some constant Q) associated to a "spin". A *configuration* $\mathbf{s} = \{s_i\}_{i\in I}$ is a prescription of a state for each atom, i.e., a function $\mathbf{s} : I \to \{1, 2, \ldots, Q\}$.



(1.6)

a general Coxeter system if a "reflection faithful representation" exists in any characteristic. If one uses a representation that is not reflection faithful, it is unknown if the character map is an isomorphim.

[5]Just consider the letter written by Pauli to Casimir just after the World War II speaking about the Ising model, a special case of the Potts model. Casimir was worried by being cut off for so long from theoretical physics of allied countries. Pauli in his reply said "nothing much of interest has happened except for Onsagers exact solution of the two-dimensional Ising model".

The partition function $Z : \mathbb{R} \to \mathbb{R}$ is a function of the parameter β (the inverse of the thermal energy) and it defines the model. One magic feature of statistical mechanics is that if you can calculate the partition function, you are mostly done: you can calculate many important observables such as the free energy, the configuration with minimal energy where the system "wants" to go, etc.

The Potts model tries to model ferromagnets (like iron). Due to physical considerations (exchange interactions tend to have short range) the partition function favor neighbors alignment. In formulas, we have

$$Z(\beta) := \sum_{\substack{\mathbf{s} \\ \text{configurations}}} \prod_{\substack{(i,j) \\ \text{neighbours}}} \exp(\beta \delta_{s_i,s_j})$$

The transfer matrix is some matrix τ satisfying the equality

$$Z(\beta) = Tr(\tau^l).$$

The most interesting feature (see [Mar08]) about writing the partition function in such a way is that all the physical information is condensed in the eigenvalues of τ

$$Tr(\tau^l) = \sum_n \lambda_n^l.$$

In the particular case of "toroidal boundary conditions", i.e. when $s_i = s_j$ if i and j are on the border of the graph and project to the same x or y component, then τ is the $Q \times Q$-matrix with diagonal $\exp(\beta)$ and off-diagonal elements equal to 1. This is a very relevant case for physics.

It is in the aim of calculating the transfer matrix (and ultimately, the partition function) that one introduces an algebra called the *transfer matrix algebra* of the model, defined by "easy" generators and some relations (we must say that there are other uses in physics for these algebras, such as identifying "equivalent" models). The transfer matrix turns out to be some specific element in the image of a representation of the transfer matrix algebra.

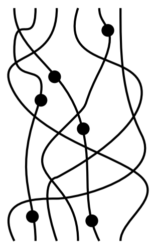
1.5. **The blob algebra.** In the case of the Potts model, its transfer matrix algebra is the usual Temperley-Lieb algebra. If one considers the Potts model but with a boundary that may have some extra degrees of freedom (for details see [GJSV13, Section 2.3]), its transfer matrix algebra is the "one boundary Temperley Lieb algebra" or "blob algebra".

The blob algebra was introduced by P. Martin and H. Saleur in [MS94] as a two-parameter diagrammatic algebra. In the same paper they constructed all the irreducible representations of that algebra when the ground field is of characteristic zero and the parameters are "good enough" (that is, when the algebra is semisimple). Some diagrammatic basis of the blob algebra turns out to be cellular in the sense of J. Graham and G. Lehrer [GL96] in any characteristic and for any choice of the parameters.

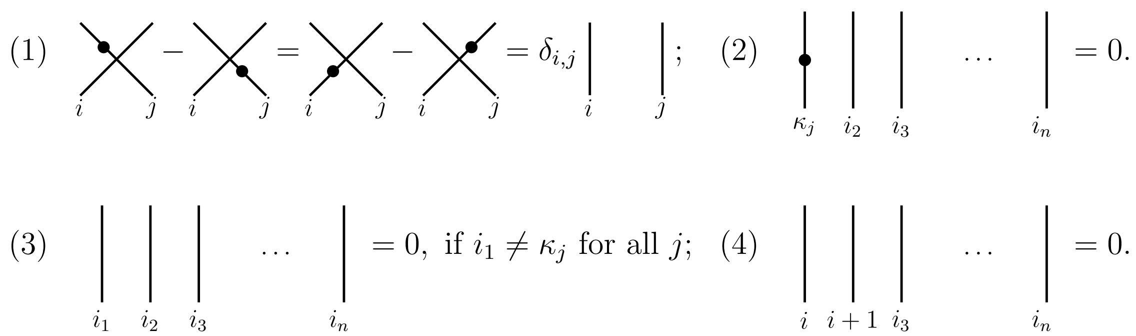
In [MW00] P. Martin and D. Woodcock completed the study of the representation theory of the blob algebra in characteristic zero by treating the "bad parameters" case. In particular they determined the decomposition numbers given by the cellular structure. In [MW03] the same authors noticed that the decomposition numbers of the blob algebra are given by KL polynomials (in Soergel's normalization) of type $\tilde{A}_1$ evaluated at 1. This remark was upgraded by the second author and S. Ryom-Hansen in [Pla13, PRH14]. In these papers it is proven that the blob algebra is graded cellular (in the sense of J. Hu and A. Mathas [HM10]) and that its graded decomposition numbers (now Laurent polynomials) are the full KL polynomials, not just evaluated at 1 (again, just for $\tilde{A}_1$). In [Bow17] C. Bowman generalized this result for generalized blob algebras by proving that these algebras are graded cellular and that its decomposition numbers are given by KL polynomials in type $\tilde{A}_n$.

On the positive characteristic side much less is known. We still know (ungraded) decomposition numbers for the blob algebra [CGM03]. Apart from this, nothing else is known. For instance, we do not have a graded counterpart of this result even in type $\tilde{A}_1$. In other words, we do not know what graded decomposition numbers for (generalized) blob algebras are. Very recently, C. Bowman and A. G. Cox [BC17] proposed a conjecture that roughly speaking says that if p is *not too small*[6] then graded decomposition numbers for the blob algebra coincide with KL polynomials in type $\tilde{A}_n$.

1.6. **The generalized blob algebra.** Let $e, n, l > 1$ be integers. Let $\kappa = (\kappa_1, \ldots, \kappa_l)$ be an l-tuple of elements of $\mathbb{Z}/e\mathbb{Z}$ "separated from each other" in the sense that for every pair $i \neq j$, the element κ_i does not belong to the set $\{\kappa_j - 1, \kappa_j, \kappa_j + 1\}$.
Consider the algebra A whose elements are $\mathbb{F}_p$-linear combinations of isotopy classes of n-string diagrams (with dots or "blobs") like the following one



where in the bottom we place some n-tuple of elements of $\mathbb{Z}/e\mathbb{Z}$ that we call *the bottom n-tuple of the diagram*. This determines in an obvious way *the top n-tuple of the diagram*. Multiplication of two diagrams is defined by vertical concatenation if the top n-tuple of the lower diagram agrees with the bottom n-tuple of the top diagram. If not, the multiplication is defined to be zero.
The *generalized blob algebra* $B^p_{l,n}(\kappa)$ is the algebra A modulo the following local relations



and two other relations reminiscent from Reidemeister II and Reidemeister III (for details see section 3).
If one does not consider relation (4), one obtains the *Cyclotomic KLR algebra* [BK09]. If one does not consider relations (3) and (4), one obtains the usual *KLR algebra* [KL09].

1.7. **One conjecture and two results.** As we said before, generalized blob algebras are graded cellular. We define $P^l_1(n)$ as the set of l-tuples of non-negative integers adding up to n. Graded cellularity of $B^p_{l,n}(\kappa)$ gives, by the general theory of graded cellular algebras and some extra effort, that $B^p_{l,n}(\kappa)$ is equipped with "graded cell modules" $\Delta^p(\boldsymbol{\lambda})$ and "graded simple

[6]In this case the authors do not provide an explicit definition of what is meant by *not to small*.

modules" $L^p(\boldsymbol{\lambda})$, for each $\boldsymbol{\lambda} \in P_1^l(n)$[7]. The *graded decomposition numbers* $d^p_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ (which against their name, are not numbers, but polynomials in $\mathbb{Z}[v,v^{-1}]$) give graded multiplicities of simple modules in composition series of cell modules. More precisely, one can define them by the formula in the Grothendieck group

$$[\Delta^p(\boldsymbol{\lambda})] = \sum_{\boldsymbol{\mu}\in P_1^l(n)} d^p_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)[L^p(\boldsymbol{\mu})]$$

Let W_l be the affine Weyl group of type $\tilde{A}_{l-1}$. We define a simply transitive action of W_l on the set of alcoves (that are some subsets of $E_l := \mathbb{R}^l/\langle(1,1,\dots,1)\rangle$). Our action, with respect to the usual one, is dilated by a factor of e and κ prescribes the distance from the origin to the walls: there is a wall in the fundamental alcove "distant"[8] $\kappa_j - \kappa_{j-1}$ from the origin, for each j (for precise definitions see section 4). We call the alcove containing the class of 0 the *fundamental alcove* A_0.
We can see an element $\boldsymbol{\lambda} \in P_1^l(n)$ as an element of $\mathbb{R}^l$ (whose coordinates add to n) and thus as an element in E_l. If the latter element belongs to the interior of an alcove, we call it *regular*. We denote by $w_{\boldsymbol{\lambda}}$ the unique element in W_l such that $\boldsymbol{\lambda} \in w_{\boldsymbol{\lambda}} \cdot A_0$. For each $w \in W_l$ there is at least one (usually many) regular $\boldsymbol{\lambda}$ such that $w = w_{\boldsymbol{\lambda}}$.

Blob vs Soergel Conjecture. *Let $\boldsymbol{\lambda},\boldsymbol{\mu} \in P_1^l(n)$ be regular, with $\boldsymbol{\mu}$ in the orbit of $\boldsymbol{\lambda}$, then*

$$h^p_{w_{\boldsymbol{\lambda}},w_{\boldsymbol{\mu}}} = d^p_{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{Z}[v,v^{-1}]$$

Remark 1.3. As we already pointed out, generalized blob algebras are quotients of cyclotomic KLR algebras. On the other hand, cyclotomic KLR algebras are isomorphic to cyclotomic Hecke algebras [BK09]. In a recent paper Elias and Losev [EL17b] showed that the decomposition numbers of cyclotomic Hecke algebras can be obtained as evaluation at 1 of certain parabolic p-Kazhdan-Lusztig polynomials. It would be tempting to believe that the previous conjecture could be deduced from Elias and Losev's work. However, this is not the case since in general cell (standard) modules for cyclotomic Hecke algebras are not the pullback of cell modules for generalized blob algebras (see [RH10]), since the cellular structure that they use (the one defined in [DJM98]) is not compatible with the one used in the generalized blob algebra.

If this conjecture is true, we obtain a new way to calculate any p-Kazhdan-Lusztig polynomial in type $\tilde{A}_{l-1}$, using generalized blob algebras. This might be very useful, mostly if one develops a better intuition of the physics related to these algebras. For this, a statistical model giving the generalized blob algebras in positive characteristic (and not only the blob algebra in characteristic zero) would be extremely interesting.
There are three reasons for us to believe in the validity of this conjecture: Theorems 1.4 and 1.5 below and some experimental checking for $l = 3$ and $p = 3$. To explain the first theorem we need some new terminology.
For each regular $\boldsymbol{\lambda} \in P_1^l(n)$, there is a combinatorial way (see section 4 for more details) to associate an n-tuple $i^{\boldsymbol{\lambda}}$ of elements in $\mathbb{Z}/e\mathbb{Z}$. Define $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$ as the subalgebra of $B^p_{l,n}(\kappa)$ consisting of $\mathbb{F}_p-$linear combinations of diagrams having $i^{\boldsymbol{\lambda}}$ as the bottom and as the top n-tuple. It is not hard to prove that $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$ is a graded cellular algebra and we denote by $\Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ the corresponding graded cell module (if $\boldsymbol{\mu}$ lies in the orbit of $\boldsymbol{\lambda}$). Then, we have

[7] In fact, the general theory tells us this only for elements $\boldsymbol{\lambda}$ in a subset of $P_1^l(n)$, but in this case one can prove that this subset is $P_1^l(n)$ itself.

[8] We mean the geodesic distance in the graph with vertices $\mathbb{Z}^l$ and where two vertices are connected by an edge if they differ by some ϵ_i.

Theorem 1.4. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P_1^l(n)$ be regular, with $\boldsymbol{\mu}$ in the orbit of $\boldsymbol{\lambda}$. Then, there is a reduced expression of $w_{\boldsymbol{\lambda}}$, that we denote by $\underline{w_{\boldsymbol{\lambda}}}$ (defined in Section 4) such that*

$$\dim_v \Delta_{\boldsymbol{\lambda}}^p(\boldsymbol{\mu}) = \sum_{l \in \mathbb{L}_{\underline{w_{\boldsymbol{\lambda}}}}(w_{\boldsymbol{\mu}})} v^{\deg(l)}. \tag{1.7}$$

As we will explain in Section 4.5, equation (1.7) relates an algorithm to calculate graded decomposition numbers for $B_{l,n}^p(\kappa)$ and an algorithm to compute p-Kazhdan-Lusztig polynomials. This theorem (valid for any rank) is half of the work needed in rank 1 to prove Blob vs Soergel conjecture (Theorem 1.5).

It goes without saying that this theorem alone is not enough to guarantee the validity of our conjecture. However, its characteristic zero analogue can be derived from it. As we have already mentioned, this fact was also proven by C. Bowman [Bow17]. Another reason to believe that Blob vs Soergel conjecture is a reasonable one is the central result of this paper:

Theorem 1.5. *Blob vs Soergel Conjecture is verified in the case $l = 2$ if $p \neq 2$.*

In mathematics, equalities between families of polynomials usually hide some deeper phenomena. We believe that Blob vs Soergel Conjecture is the combinatorial shadow of an equivalence of "Hecke categories".

1.8. **Upgrading Blob vs Soergel conjecture.** Let us fix the data for a generalized blob algebra, i.e., (e, l, n, κ), and $\boldsymbol{\lambda} \in P_1^l(n)$ regular. Define $\mathcal{H}_{BS}^l(\leq \boldsymbol{\lambda})$ as the full subcategory of $\mathbb{F}_p \otimes_R \mathcal{H}_{BS}(W_l)$ having as objects all $BS(\underline{w_{\boldsymbol{\mu}}})$ where $\boldsymbol{\mu}$ is in the orbit of $\boldsymbol{\lambda}$ and $w_{\boldsymbol{\mu}} \leq w_{\boldsymbol{\lambda}}$ in the Bruhat order. It is clear that this category has a finite number of objects.

On the other hand, define $\mathrm{Blob}^l(\leq \boldsymbol{\lambda})$ as the category with objects all elements $\boldsymbol{\mu} \in P_1^l(n)$ in the orbit of $\boldsymbol{\lambda}$ such that $w_{\boldsymbol{\mu}} \leq w_{\boldsymbol{\lambda}}$. Given two objects $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $\mathrm{Blob}^l(\leq \boldsymbol{\lambda})$, the morphism space from $\boldsymbol{\mu}$ to $\boldsymbol{\nu}$ is the space $B_{l,n}^{p,\boldsymbol{\mu},\boldsymbol{\nu}}$, defined as the subspace of $B_{l,n}^p$ with bottom n-tuple given by $i^{\boldsymbol{\mu}}$ and top n-tuple given by $i^{\boldsymbol{\nu}}$. The composition of morphisms is just multiplication in $B_{l,n}^p$.

Categorical Blob vs Soergel Conjecture. *For each integer $l \geq 2$, there is an equivalence of categories*

$$\mathrm{Blob}^l(\leq \boldsymbol{\lambda}) \cong \mathcal{H}_{BS}^l(\leq \boldsymbol{\lambda})$$

sending $\boldsymbol{\mu}$ to $BS(\underline{w_{\boldsymbol{\mu}}})$.

Suppose that this conjecture is verified. Consider some infinite sequence $(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \boldsymbol{\lambda}_3, \ldots)$ of elements $\boldsymbol{\lambda}_i \in P_1^l(n_i)$ in the same orbit, with $n_1 < n_2 < n_3 < \ldots$ and $w_{\boldsymbol{\lambda}_1} < w_{\boldsymbol{\lambda}_2} < w_{\boldsymbol{\lambda}_3} < \ldots$. This gives a chain of inclusions[9]

$$\mathrm{Blob}^l(\leq \boldsymbol{\lambda}_1) \subset \mathrm{Blob}^l(\leq \boldsymbol{\lambda}_2) \subset \mathrm{Blob}^l(\leq \boldsymbol{\lambda}_3) \subset \cdots .$$

Define

$$\mathrm{Blob}^l(\infty) = \varinjlim \mathrm{Blob}^l(\leq \boldsymbol{\lambda}_i),$$

as the direct limit of this chain. Define in a similar fashion

$$\mathcal{H}_{BS}^{l,\infty} = \varinjlim \mathcal{H}_{BS}^l(\leq \boldsymbol{\lambda}_i).$$

By general properties of direct limits, the Categorical Blob vs Soergel conjecture implies an equivalence of categories

$$\mathrm{Blob}^l(\infty) \cong \mathcal{H}_{BS}^{l,\infty}. \tag{1.8}$$

[9]More precisely, for each i, there is a fully-faithful functor $F_i : \mathrm{Blob}^l(\leq \boldsymbol{\lambda}_i) \to \mathrm{Blob}^l(\leq \boldsymbol{\lambda}_{i+1})$ which maps an object $\boldsymbol{\mu}$ of $\mathrm{Blob}^l(\leq \boldsymbol{\lambda}_i)$ (and therefore in $P_1^l(n_i)$) to the object $\boldsymbol{\mu}'$ of $\mathrm{Blob}^l(\leq \boldsymbol{\lambda}_{i+1})$ (and therefore in $P_1^l(n_{i+1})$) which is obtained from $\boldsymbol{\mu}$ by adding $(n_{i+1} - n_i)/l$ to each component.

This equivalence, of course, is preserved if one takes the additive closure (objects in the additive closure are formal finite direct sums of objects in the original category)

$$\mathrm{Blob}^l(\infty)_\oplus \cong (\mathcal{H}^{l,\infty}_{BS})_\oplus. \tag{1.9}$$

The difference between $(\mathcal{H}^{l,\infty}_{BS})_\oplus$ and $\mathbb{F}_p \otimes_R \mathcal{H}_{BS}(W_l)$ is that in the former category there is only one Bott-Samelson object for each element of the group. This difference clearly disappears when one passes to Karoubian envelopes. In other words, the corresponding Karoubian envelopes give equivalent categories:

$$(\mathcal{H}^{l,\infty}_{BS})^e_\oplus \cong (\mathbb{F}_p \otimes_R \mathcal{H}_{BS}(W_l))^e. \tag{1.10}$$

Let us define the *Blob category* Blob^l by the equality

$$\mathrm{Blob}^l := \mathrm{Blob}^l(\infty)^e_\oplus.$$

The equivalence 1.10, together with the equivalence 1.9, would give us the following

Main consequence of the Categorical Blob vs Soergel Conjecture. *For each integer $l > 1$, there is an equivalence of categories*

$$\mathrm{Blob}^l \cong \mathbb{F}_p \otimes_R \mathcal{H}(W_l)$$

In principle, the category Blob^l should depend on the descending chain $\boldsymbol{\lambda}_i$ chosen, but if the Categorical Blob vs Soergel Conjecture is verified, this category would be (modulo equivalence) independent of this choice. This equivalence (if verified) is probably related to the Categorical Schur-Weyl duality, which is implicit in [Rou08] and a version of it is explained in [RW15, Theorem 8.1]. If this equivalence of categories is verified, one would certainly have new insights into the anti-spherical category (a quotient of $\mathcal{H}$) by the geometry defining the Blob category.

If verified, the Categorical Blob vs Soergel conjecture would give, when considering the endomorphism spaces of one object, an isomorphism of $\mathbb{F}_p$-algebras

$$\mathbb{F}_p \otimes_R \mathrm{End}_{\mathcal{H}}(BS(\underline{w}_{\boldsymbol{\lambda}})) \cong B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa). \tag{1.11}$$

As an evidence in favor of this isomorphism, we noticed that $\mathbb{F}_p \otimes_R \mathrm{End}_{\mathcal{H}}(BS(\underline{w}_{\boldsymbol{\lambda}})) \cong B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$ as graded vector spaces . A fact that can be easily deduced from Theorem 1.4.

1.9. **Organization of the paper.** In section 2 we introduce the Hecke algebra and the Hecke category as well as intersection forms and the p-canonical basis (using diagrams). In section 3 we introduce generalized blob algebras and explain their graded cellular algebra structure. We also explain the combinatorial way to associate $\boldsymbol{\lambda} \rightsquigarrow \mathfrak{t}^{\boldsymbol{\lambda}} \rightsquigarrow \boldsymbol{i}^{\boldsymbol{\lambda}}$ and introduce the algebra $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$. In section 4 we study some an alcove geometry in order to find the map $\boldsymbol{\lambda} \rightarrow \underline{w}_{\boldsymbol{\lambda}}$ and prove Theorem 1.4. Finally, in section 5 we prove Theorem 1.5.

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2. The p-canonical basis via Soergel calculus

2.1. **The Hecke algebra.** Let (W,S) be a Coxeter system and $(m_{sr})_{s,r\in S}$ its Coxeter matrix. Let $l: W \to \mathbb{N}$ be the corresponding length function and $\leq$ the Bruhat order on W. Consider the ring $\mathcal{L} = \mathbb{Z}[v^{\pm 1}]$ of Laurent polynomials with integer coefficients in one variable v.
The *Hecke algebra* $\mathcal{H} = \mathcal{H}(W,S)$ of a Coxeter system (W,S) is the associative algebra over $\mathcal{L}$ with generators $\{H_s\}_{s\in S}$, quadratic relations $(H_s+v)(H_s-v^{-1})=0$ for all $s\in S$, and for every couple $s,r\in S$, braid relations

$$H_sH_rH_s\cdots = H_rH_sH_r\cdots$$

with m_{sr} terms on each side of the equation.
Consider $x \in W$. To any $sr\cdots t$ reduced expression of x one can associate the element $H_sH_r\cdots H_t \in \mathcal{H}$. H. Matsumoto proved that this element, that we call H_x, is independent of the choice of reduced expression of x. N. Iwahori proved that

$$\mathcal{H} = \bigoplus_{x\in W} \mathcal{L}H_x.$$

2.2. **Kazhdan-Lusztig polynomials.** Let us define for $s\in S$, the element $\underline{H}_s := H_s + v$. There is a unique ring homomorphism (moreover, an involution) $H \mapsto \overline{H}$ on H such that $\overline{v} = v^{-1}$ and $\overline{H_x} = (H_{x^{-1}})^{-1}$. We will call an element *self-dual* if it is invariant under $\overline{(-)}$. The fact that for every element $x\in W$ there is a unique self-dual element $\underline{H}_x \in H$, such that $\underline{H}_x \in H_x + \sum_{y\in W} v\mathbb{Z}[v]H_y$, was proved by D. Kazhdan and G. Lusztig in [KL79]. We call the set $\{\underline{H}_x\}_{x\in W}$ the *Kazhdan-Lusztig basis* or the *Canonical basis* of $\mathcal{H}$. It is an basis of the Hecke algebra as an $\mathcal{L}$-module. In formulas,

$$\mathcal{H} = \bigoplus_{x\in W} \mathcal{L}\underline{H}_x.$$

For each couple of elements $x,y\in W$ we define $h_{y,x}\in\mathcal{L}$ by the formula

$$\underline{H}_x = \sum_y h_{y,x}H_y.$$

Remark 2.1. *The* Kazhdan-Lusztig polynomials *(as defined in* [KL79]*) are given by the formula*

$$P_{y,x} = (v^{l(y)} - v^{l(x)})h_{y,x},$$

and they are polynomials in $q := v^{-2}$.

Definition 2.2. *Let* $\underline{w} = sr\cdots t$ *be a reduced expression of* $w\in W$. *Then we define*

$$H_{\underline{w}} := \underline{H}_s\underline{H}_t\cdots\underline{H}_t.$$

For Weyl groups or affine Weyl groups and for a given prime number p, there is another family of bases of the Hecke algebra, the so-called p-canonical basis $\underline{H}^p_x$

$$\mathcal{H} = \bigoplus_{x\in W} \mathcal{L}\underline{H}^p_x.$$

Until this date there is no algebraic way to calculate the p-canonical basis, all the known ways rely on categorical calculations. We will define the diagrammatic Hecke Category of Elias and Williamson, that, for many purposes (in particular to prove our main theorem 1.5 in Section 5) is the best way to calculate the p-canonical basis. From now on we will concentrate on the case of affine Weyl groups of type A; we consider for the rest of the paper $W = \tilde{A}_n$.

2.3. **Realizations.** Recall that a realization, as defined in [EW16, §3.1]) consists of a commutative ring $\Bbbk$ and a free and finitely generated $\Bbbk$-module $\mathfrak{h}$ together with subsets

$$\{\alpha_s\}_{s\in S} \subset \mathfrak{h}^* \quad \text{and} \quad \{\alpha_s^\vee\}_{s\in S} \subset \mathfrak{h}$$

of "roots" and "coroots" such that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$ for all $s \in S$ and such that the formulas

$$s(v) := v - \langle \alpha_s, v\rangle \alpha_s^\vee \quad \text{for } s \in S \text{ and } v \in \mathfrak{h},$$

define an action of W on $\mathfrak{h}$.
Unless otherwise stated we will assume in this paper that $\Bbbk = \mathbb{F}_p$ and that $\mathfrak{h}$ is the Cartan matrix representation of W, i.e., $\mathfrak{h} = \bigoplus_{s\in S} \Bbbk\alpha_s^\vee$ and the elements $\{\alpha_s\} \subset \mathfrak{h}^*$ are defined by the equations

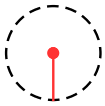
$$\langle \alpha_t^\vee, \alpha_s \rangle = -2\cos(\pi/m_{st}) \tag{2.1}$$

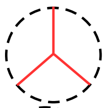
(by convention $m_{ss} = 1$ and $\pi/\infty = 0$).
Let $R = S(\mathfrak{h}^*)$ be the ring of regular functions on $\mathfrak{h}$ or, equivalently, the symmetric algebra of $\mathfrak{h}^*$ over $\Bbbk$. We look at R as a graded $\Bbbk$-algebra by declaring $\deg \mathfrak{h}^* = 2$. The action of W on $\mathfrak{h}^*$, extends to an action on R, by functoriality. For $s \in S$, let $\partial_s : R \to R[-2]$ be the *Demazure operator* defined by

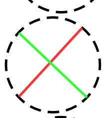
$$\partial_s(f) = \frac{f - sf}{\alpha_s}.$$

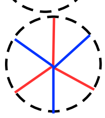
In [EW16, §3.3] the authors prove that this operator is well defined under our assumptions.

2.4. **Towards the morphisms in $\mathcal{H}_{\mathrm{BS}}$.** An *S-graph* is a finite, planar, decorated graph with boundary properly embedded in the planar strip $\mathbb{R} \times [0,1]$. Each edge is colored by an element of S. The vertices in this graph can be of 3 types:

(1) univalent vertices ("dots"): 

(2) 3-valent vertices: 

(3) 4-valent vertices:  , here we have $m_{r,g} = 2$.

(4) 6-valent vertices:  , here we have $m_{r,b} = 3$.

Additionally any S-graph may have its regions (the connected components of the complement of the graph in $\mathbb{R} \times [0,1]$) decorated by boxes containing homogenous elements of R.
The following is an example of an S-graph with $m_{b,r} = 3$, $m_{b,g} = 2$, $m_{g,r} = 3$:

$$\text{Aqui va un example de Soergel.} \tag{2.2}$$

where f and g are homogeneous polynomials in R.
A Soergel graph has a *degree*, which accumulates $+1$ for every dot, 1 for every trivalent vertex, and the degree of each polynomial
For example, the degree of the S-graph above is

$$-1 + 1 - 1 + 1 - 1 - 1 + 1 + 1 + \deg f + \deg g = \deg f + \deg g.$$

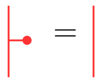
The intersection of an S-graph with $\mathbb{R} \times \{0\}$ (resp. with $\mathbb{R} \times \{1\}$) is a sequence of colored points called *bottom boundary* (resp. *top boundary*). In our example, the bottom (resp. top) boundary of the S-graph is (b, r, r, b, g, r) (resp. (r, g, b, r, g, g)).

2.5. **Relations in $\mathcal{H}_{\mathrm{BS}}$.** We define the *Bott-Samelson diagrammatic category* $\mathcal{H}_{\mathrm{BS}}$ as the monoidal category where objects are sequences $\underline{w}$ in S. If $\underline{x}$ and $\underline{y}$ are two such sequences, we define $\mathrm{Hom}_{\mathcal{H}_{\mathrm{BS}}}(\underline{x}, \underline{y})$ as the free R-module generated by isotopy classes of S-graphs with bottom boundary $\underline{x}$ and top boundary $\underline{y}$, modulo the local relations below. Hom spaces are graded by the degree of the graphs, because all the relations below are homogeneous. The structure of this monoidal category is given by horizontal and vertical concatenation of diagrams.

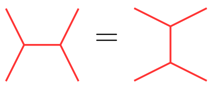
In what follows, the rank of a relation is the number of colors involved in the relation. We use the color red for r, blue for b and green for g.

2.5.1. ***Rank 1 relations.***

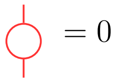
- *Frobenius unit:*


(2.3)

- *Frobenius associativity:*

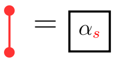

(2.4)

- *Needle relation:*



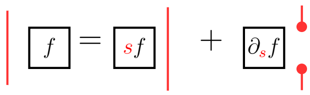
(2.5)

- *Barbell relation:*



(2.6)

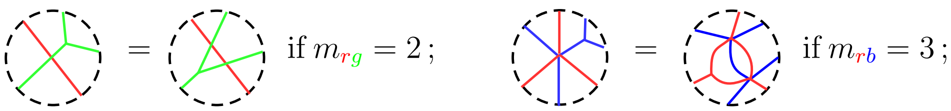
- *Nil Hecke relation:*



(2.7)

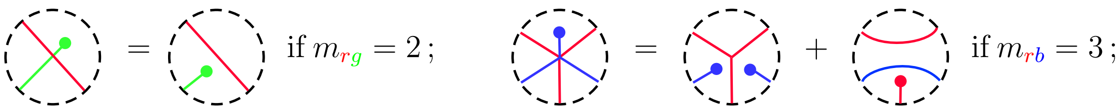
2.5.2. ***Rank 2 relations.***

- *Two-color associativity:*



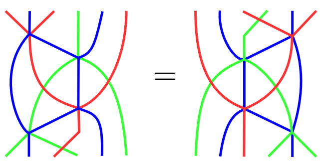
(2.8)

- *Elias' Jones-Wenzl relation:*



(2.9)

2.5.3. ***Rank 3 relations.*** For every triple of simple reflections r, b, g such that they form an A_3-root system (i.e., $m_{r,b} = m_{b,g} = 3$ and $m_{r,g} = 2$) we have the following *Zamolodchikov*relation:


(2.10)

This ends the definition of $\mathcal{H}_{\mathrm{BS}}$.

2.6. **The Hecke category $\mathcal{H}$.** If $M = \bigoplus_i M^i$ is a $\mathbb{Z}$-graded object, we denote by $M(1)$ its grading shift defined by $M(1)^i = M^{i+1}$. For an additive category $\mathcal{A}$ we denote by $[\mathcal{A}]$ its split Grothendieck group. If in addition $\mathcal{A}$ has Hom spaces enriched in graded vector spaces we denote by $\mathcal{A}^{\oplus}$ its additive graded closure, i.e. objects are formal finite direct sums $\bigoplus a_i(m_i)$ for certain objects $a_i \in \mathcal{A}$ and "grading shifts" $m_i \in \mathbb{Z}$ and

$$\mathrm{Hom}_{\mathcal{A}^{\oplus}}(\bigoplus a_i(m_i), \bigoplus b_j(n_j)) := \bigoplus \mathrm{Hom}(a_i, b_j)(n_j - m_i).$$

The category $\mathcal{A}^{\oplus}$ is equipped with a grading shift functor defined on objects by $\bigoplus a_i(m_i) \mapsto \bigoplus a_i(m_i + 1)$. We define $\mathcal{A}^e$ to be the Karoubian envelope of $\mathcal{A}^{\oplus}$. We can finally define the *Hecke category* $\mathcal{H} := \mathcal{H}^e_{\mathrm{BS}}$.

2.7. **Light leaves.** Let $\underline{x} = s_1 s_2 \dots s_m$ be an expression (a sequence of elements in S). A *subsequence* of $\underline{x} = s_1 s_2 \dots s_m$ is a sequence $\pi_1 \pi_2 \dots \pi_m$ such that $\pi_i \in \{e, s_i\}$ for all $1 \leq i \leq m$. Instead of working with subsequences, we work with the equivalent datum of a sequence $\mathbf{e} = \mathbf{e}_1 \mathbf{e}_2 \dots \mathbf{e}_m$ of 1's and 0's giving the indicator function of a subsequence, which we refer to as a *01-sequence*. Associated to this, one has the *Bruhat stroll*. It is the sequence $x_0, x_1, \dots, x_m$ defined inductively by $x_0 = e$ and

$$x_i := s_1^{\mathbf{e}_1} s_2^{\mathbf{e}_2} \dots s_i^{\mathbf{e}_i}$$

for $1 \leq i \leq m$. We call x_i the i^{th}-*point* and x_m the *end-point* of the Bruhat stroll. We denote the end-point by $\underline{x}^{\mathbf{e}}$.

Light leaves and *Double leaves* for Soergel bimodules were introduced in [Lib08b] and [Lib15]. They give bases, as R-modules of the Hom spaces between Bott-Samelson bimodules. We advice [Lib17, § 6.4–6.5] in order to get used to these combinatorial objects (in that paper the dot is called m_s, the trivalent vertex is called j_s and the four and six valent vertices are called f_{sr}). We also advice to read [EW16, § 6.1–6.3], where these bases are explained in the diagrammatic language used in this paper.

We call $\mathbb{L}_{\underline{w}}(x)$ (resp. $\mathbb{L}^d_{\underline{w}}(x)$) the set of light leaves (resp. light leaves of degree d) with source $\underline{w}$ and target some expression of $x \in W$. In particular, the cardinality of $\mathbb{L}_{\underline{w}}(x)$ is equal to the number of 01-sequences with end-point equal to x.

Let X be a set of homogeneous elements of graded vector spaces. We define the *degree* of X by the formula

$$d(X) = \sum_{x \in X} v^{-\deg(x)}.$$

In [Lib15, § 5.2] the following is proved

$$H_{\underline{w}} = \sum_{x \leq w} d(\mathbb{L}_{\underline{w}}(x)) H_x. \tag{2.11}$$

2.8. **Intersection forms.** In this section $\Bbbk = \mathbb{Z}$ and d is any integer. If n and m are such that $\mathbb{L}^d_{\underline{w}}(x) = \{l_1^+, \dots, l_n^+\}$ and

$$\mathbb{L}^{-d}_{\underline{w}}(x) = \{l_1^-, \dots, l_m^-\}.$$

Expand the element $l_j^- \circ (l_i^+)^a \in \mathrm{End}(\underline{x})$ in the double leaves basis:

$$l_j^- \circ (l_i^+)^a = n_{ij}\mathrm{id} + \sum_k ll_k^a \cdot ll'_k,$$

where ll_k, ll'_k belong to some $\mathbb{L}_{\underline{w}}(y_k)$ with $y_k < x$. Define the *d-th grading piece of the intersection form* as the matrix $I_{\underline{w},x,d} = (n_{ij})_{i,j}$. Finally, we define the intersection form simply as the matrix direct sum of the degree pieces

$$I_{\underline{w},x} := \bigoplus_d I_{\underline{w},x,d}.$$

If p is a prime, the matrix $I^p_{\underline{w},x}$ is the reduction of the matrix $I_{\underline{w},x}$ modulo p. In [JW17] the following formula is proven

$$H_{\underline{w}} = \sum_{y\leq w} Grk\left(I^p_{\underline{w},y}\right)\underline{H}^p_y. \tag{2.12}$$

In [EW16, Definition 6.24] the authors define a character map ch : $[\mathcal{H}] \to H$ and in [EW16, Corollary 6.27] they prove that it is an isomorphism sending $[B_s]$ to $\underline{H}_s$ and $[v]$ to the empty word shifted by 1. This is the reason to call $\mathcal{H}$ the Hecke category.

2.9. **p-canonical basis.** Following Soergel's classification of indecomposable Soergel bimodules, in [EW16, Theorem 6.26] the authors prove that the indecomposable objects in $\mathcal{H}$ are indexed by W modulo shift, and they call B_w the indecomposable object corresponding to $w \in W$. It happens that the object B_s is the sequence with one element $(s) \in \mathcal{H}$. Because of this, if $\underline{w} = (s, r, \cdots, t)$ we will sometimes denote by $BS(\underline{w}) := B_sB_r\cdots B_t$ the element $\underline{w} \in \mathcal{H}$ (this is also common usage for Soergel bimodules).
In [EW14] the authors prove that, if $\Bbbk$ is the field $\mathbb{R}$, then $\mathrm{ch}([B_w]) = \underline{H}_w$. (More precisely, to obtain this result one must combine the equivalence between $\mathcal{H}$ and the category of Soergel bimodules proved in [EW16] with the main results of [EW14] and [Lib08a].) Thus the indecomposable objects in $\mathcal{H}$ categorify the Kazhdan-Lusztig basis when we work over the real numbers. But when $\Bbbk$ is the field $\mathbb{F}_p$, then the character of B_w is no longer forced to be the canonical basis.

Notation 2.3. *Let $\Bbbk = \mathbb{F}_p$. To emphasize the field in which we are working on, we will call B^p_w the indecomposable objects of $\mathcal{H}$ (instead of the usual B_w). We use the notation* $\mathrm{ch}([B^p_w]) = \underline{H}^p_w \in \mathcal{H}$. *The* p-canonical basis *is the set $\{\underline{H}^p_w\}_{w\in W}$, which can be proved to be a basis of the Hecke algebra. If we write the p-canonical basis in terms of the standard basis*

$$\underline{H}^p_y = \sum_{x\leq y} h^p_{x,y}H_x, \tag{2.13}$$

the polynomials $h^p_{x,y}$ are called p-Kazhdan-Lusztig polynomials.

2.10. **An important formula.** By (2.11), (2.12) and (2.13) we obtain

$$\begin{aligned}\sum_{x\leq w} d(\mathbb{L}_{\underline{w}}(x))H_x &= \sum_{y\leq w} Grk\left(I^p_{\underline{w},y}\right)\sum_{x\leq y} h^p_{x,y}H_x \\ &= \sum_{x\leq w}\left(\sum_{x\leq y\leq w} Grk\left(I^p_{\underline{w},y}\right)h^p_{x,y}\right)H_x.\end{aligned} \tag{2.14}$$

By equating the coefficients we obtain, for each $x \leq w$ the equation

$$d(\mathbb{L}_{\underline{w}}(x)) = \sum_{x\leq y\leq w} Grk\left(I^p_{\underline{w},y}\right)h^p_{x,y}. \tag{2.15}$$

As $h^p_{y,y} = Grk\left(I^p_{\underline{w},w}\right) = 1$, for all $y \in W$ and any reduced expression $\underline{w}$ of w, by expanding and rearranging the terms of Equation 2.15, we obtain

$$d(\mathbb{L}_{\underline{w}}(x)) - \sum_{x<y<w} Grk\left(I^p_{\underline{w},y}\right)h^p_{x,y} = Grk\left(I^p_{\underline{w},x}\right) + h^p_{x,w}. \tag{2.16}$$

The term $d(\mathbb{L}_{\underline{w}})$ is obtained by a simple calculation in the Hecke algebra, so we assume that we always know it.
Let us suppose for a moment that $p = 0$. As $Grk\left(I^0_{\underline{w},y}\right)$ is self-dual under the involution $\overline{(-)} : v \mapsto v^{-1}$ and $h^0_{x,w} = h_{x,w} \in v\mathbb{Z}[v]$, if we know the left-hand side of Equation 2.16, we can easily calculate both $Grk\left(I^0_{\underline{w},y}\right)$ and $h^0_{x,w}$. So by induction on $l(w) - l(x)$ one can obtain

all Kazhdan-Lusztig polynomials and all the graded ranks of the intersection forms with this algorithm.
In positive characteristic things don't work as smoothly as in characteristic zero. But the $\tilde{A}_1$ case is still nice, because in that case $h^p_{x,y} \in \mathbb{Z}[v]$[10]. This inclusion can be seen by the fact that in that case there exist no light leaves of negative degree [EL17a, Corollary 3.8]. This inclusion implies that if one knows $h^p_{x,y}(0)$ for all pair x, y, one can calculate as before all the p-Kazhdan-Lusztig polynomials recursively. A closed formula for $h^p_{x,y}(0)$ is given in [JW17, Lemma 5.1]. In order to introduce such a formula we need a bit more of notation. In type $\tilde{A}_1$ the corresponding affine Weyl group is the infinite dihedral group $W = \langle s, t \mid s^2 = t^2 = e\rangle$. In this group, each element different from the identity has a unique reduced expression of the form

$$k_s := sts\dots \ (k\text{-terms}) \qquad \text{and} \qquad k_t := tst\dots \ (k\text{-terms}), \tag{2.17}$$

for some integer $k \geq 1$. We use the convention $0_s = 0_t = e$.
Given two non-negative integers a and b we write their p-adic decomposition as

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_r p^r \qquad \text{and} \qquad b = b_0 + b_1 p + b_2 p^2 + \cdots + b_s p^s, \tag{2.18}$$

where $0 \leq a_i < p$, $0 \leq b_i < p$, $a_r \neq 0$ and $b_s \neq 0$. Then we say that a contains b to base p if $s < r$ and $b_i = 0$ or $b_i = a_i$, for all $0 \leq i \leq s$. We then define the function $f_p : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \{0, 1\}$ given by

$$f_p(a,b) = \begin{cases} 1, & \text{if } a+1 \text{ contains } b \text{ to base } p; \\ 0, & \text{otherwise.} \end{cases} \tag{2.19}$$

Let $y = k_s \in W$, for some $k \geq 1$. Then, we have

$$h^p_{x,y}(0) = \begin{cases} f_p(k-1, j), & \text{if } x = (k-2j)_s \text{ for some } 0 \leq j \leq \lceil \frac{k-2}{2} \rceil; \\ 0, & \text{otherwise.} \end{cases} \tag{2.20}$$

Of course, equation (2.20) remains hold if we replace s by t.

3. Generalized blob algebras are graded cellular

We will give two equivalent versions of these algebras. Both of them will be needed in the sequel.

3.1. **Generalized blob algebra, algebraic definition.** From now on, fix integers $e, l, n > 1$ and set $I_e = \mathbb{Z}/e\mathbb{Z}$. We refer to the elements of I_e^l as *multicharges*. An *adjacency-free multicharge* $\kappa = (\kappa_1, \dots, \kappa_l)$ is a multicharge such that $\kappa_i \notin \{\kappa_j, \kappa_j + 1, \kappa_j - 1\}$ for all $i \neq j$. For such a multicharge to exist, the inequality $e \geq 2l$ has to hold. Given $i \in I_e$ we define

$$\langle i \,|\, \kappa \rangle = |\{\, j \,|\, 1 \leq j \leq l,\, \kappa_j = i \,\}|. \tag{3.1}$$

This number if always 0 or 1 if κ is an adjacency-fee multicharge.

Definition 3.1. *Given integers $e, l, n > 1$ and an adjacecy-free multicharge $\kappa \in I_e^l$ the* generalized blob algebra $B^p_{l,n}(\kappa)$ of level l on n strings *is defined to be the unital, associative $\mathbb{F}_p$-algebra with generators*

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{e(\boldsymbol{i}) \mid \boldsymbol{i} \in I_e^n\}$$

and relations

[10] This inclusion is not even true in finite type A, as proved recently in [LW17].

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}), \tag{3.2}$$
$$\sum_{\boldsymbol{i}\in I_e^n} e(\boldsymbol{i}) = 1, \tag{3.3}$$
$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i}) y_r, \tag{3.4}$$
$$\psi_r e(\boldsymbol{i}) = e(s_r \boldsymbol{i})\psi_r, \tag{3.5}$$
$$y_r y_s = y_s y_r, \tag{3.6}$$
$$\psi_r y_s = y_s \psi_r, \qquad \textit{if } s \neq r, r+1 \tag{3.7}$$
$$\psi_r \psi_s = \psi_s \psi_r, \qquad \textit{if } |s-r|>1 \tag{3.8}$$
$$\psi_r y_{r+1} e(\boldsymbol{i}) = (y_r \psi_r - \delta_{i_r, i_{r+1}}) e(\boldsymbol{i}) \tag{3.9}$$
$$y_{r+1} \psi_r e(\boldsymbol{i}) = (\psi_r y_r - \delta_{i_r, i_{r+1}}) e(\boldsymbol{i}) \tag{3.10}$$
$$\psi_r^2 e(\boldsymbol{i}) = \begin{cases} 0 & \textit{if } i_r = i_{r+1} \\ e(\boldsymbol{i}) & \textit{if } i_r \neq, i_{r+1}, i_{r+1} \pm 1 \\ (y_{r+1} - y_r) e(\boldsymbol{i}) & \textit{if } i_{r+1} = i_r + 1 \\ (y_r - y_{r+1}) e(\boldsymbol{i}) & \textit{if } i_{r+1} = i_r - 1 \end{cases} \tag{3.11}$$
$$\psi_r \psi_{r+1} \psi_r e(\boldsymbol{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} - 1) e(\boldsymbol{i}) & \textit{if } i_{r+2} = i_r = i_{r+1} - 1 \\ (\psi_{r+1}\psi_r\psi_{r+1} + 1) e(\boldsymbol{i}) & \textit{if } i_{r+2} = i_r = i_{r+1} + 1 \\ (\psi_{r+1}\psi_r\psi_{r+1}) e(\boldsymbol{i}) & \textit{otherwise} \end{cases} \tag{3.12}$$
$$y_1^{\langle i_1 | \kappa \rangle} e(\boldsymbol{i}) = 0 \tag{3.13}$$
$$e(\boldsymbol{i}) = 0 \qquad \textit{if } i_2 = i_1 + 1. \tag{3.14}$$

where for a sequence $\boldsymbol{i} \in I_e^n$, $i_j \in I_e$ denotes the j'th coordinate of $\boldsymbol{i}$.

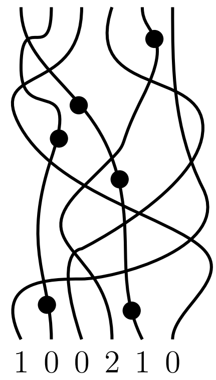
We consider $B_{l,n}^p(\kappa)$ as a graded algebra by decreeing

$$deg\ e(\boldsymbol{i}) = 0, \qquad deg\ y_r = 2, \qquad deg\ \psi_s e(\boldsymbol{i}) = \begin{cases} -2, & \textit{if } i_s = i_{s+1} \\ 1, & \textit{if } i_s = i_{s+1} \pm 1 \\ 0, & \textit{otherwise} . \end{cases}$$

Remark 3.2. *Bowman defined in* [Bow17, Definition 7.1] *a family of algebras $KL_{l,h,n}(\kappa)$ which are a one parameter generalization of the Generalized blob algebras. More precisely, $B_{l,n}^p(\kappa)$ is the specialization of $KL_{l,h,n}(\kappa)$ at $h = 1$.*

3.2. **Generalized blob algebras, diagrammatic definition.** We will explain a way to "draw" the elements of $B_{l,n}^p(\kappa)$. This can be achieved by using a variant of the diagrammatic calculus of Khovanov and Lauda [KL09].

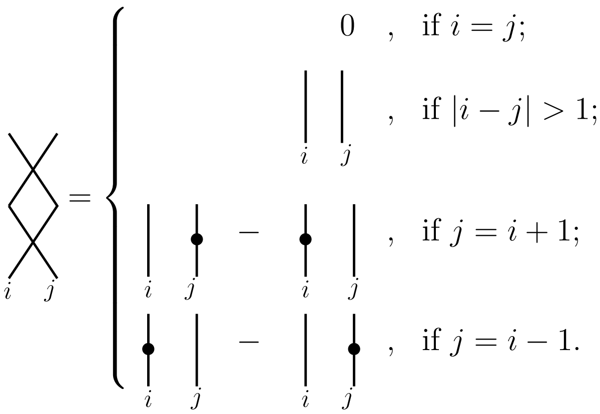
A *Khovanov-Lauda diagram on n-strings* (or simply a diagram when no confusion is possible) consists of n points on each of two parallel edges (the top edge and bottom edge) and n arcs connecting the n points in one edge with the n points on the other edge. Arcs can intersect, but no triple intersections are allowed. Each arc can be decorated by a finite number of dots, but dots cannot be located on an intersection of two arcs. Finally, each diagram is labelled by a sequence $\boldsymbol{i} = (i_1, \ldots, i_n) \in I_e^n$, which is written bellow the bottom edge. An example of such diagrams is depicted below.



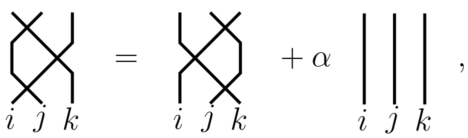
(3.15)

Given a diagram D we denote by $b(D)$ the bottom sequence labelling D (read from left to right). This sequence determines, in the obvious fashion, a top sequence, which we denote by $t(D)$. For example, if D is the diagram in (3.15) then $t(D) = (1, 0, 0, 0, 2, 1)$.
Let us define the diagrammatic algebra $B^p_{l,n}(\kappa)'$. As a vector space it consists of the $\mathbb{F}_p-$linear combinations of Khovanov-Lauda diagrams on n-strings modulo planar isotopy and modulo the following relations:

(3.16)

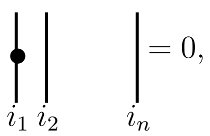


(3.17)

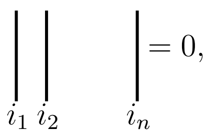


(3.18)

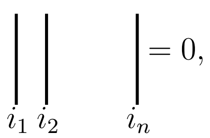
where $\alpha = -1$ if $i = k = j - 1$, $\alpha = 1$ if $i = k = j + 1$ and $\alpha = 0$ otherwise.



if $i_1 = \kappa_j$ for some $1 \le j \le l$;



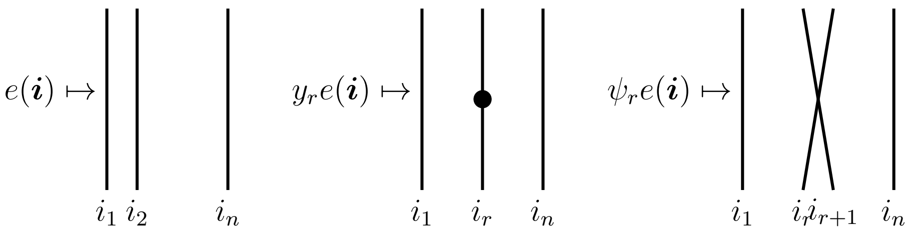
if $i_1 \ne \kappa_j$ for all $1 \le j \le l$;



if $i_2 = i_1 + 1$.

The multiplication DD' between two diagrams D and D' in $B^p_{l,n}(\kappa)'$ is defined by vertical concatenation (D on top of D') when $b(D) = t(D')$. If this last equality is not satisfied, the product is defined to be zero. We extend this product to all of $B^p_{l,n}(\kappa)'$ by linearity. With this we end the definition of $B^p_{l,n}(\kappa)'$

An isomorphism between the algebraic version $B^p_{l,n}(\kappa)$ and the diagrammatic version $B^p_{l,n}(\kappa)'$ of the generalized blob algebra is given by



(3.19)

3.3. **Graded cellular algebras.** We briefly recall the theory of graded cellular algebras, which were defined by Hu and Mathas [HM10] as a graded version of Graham and Lehrer's cellular algebras [GL96].

Definition 3.3. *Let A be an associative, $\mathbb{Z}$-graded and finite-dimensional $\mathbb{F}_p$-algebra. Then, A is called a graded cellular algebra with graded cell datum $(\Lambda, T, C, \deg)$ if the following conditions are satisfied:*

(a) *$(\Lambda, \geq)$ is a finite poset. For each $\lambda \in \Lambda$ there is a finite set $T(\lambda)$. Furthermore, the algebra A has an $\mathbb{F}_p$-basis $\mathcal{C} = \{c^\lambda_{\mathfrak{st}} \mid \lambda \in \Lambda,\ \mathfrak{s}, \mathfrak{t} \in T(\lambda)\}$.*
(b) *The $\mathbb{F}_p$-linear map $*$ determined by $(c^\lambda_{\mathfrak{st}})^* = c^\lambda_{\mathfrak{ts}}$ is an $\mathbb{F}_p$-algebra anti-isomorphism of A.*
(c) *Let $\lambda \in \Lambda$, $\mathfrak{s}, \mathfrak{t} \in T(\lambda)$ and $a \in A$. Then, there are scalars $r_{\mathfrak{sv}}(a) \in \mathbb{F}_p$ (not depending on $\mathfrak{t}$) such that*
$$ac^\lambda_{\mathfrak{st}} \equiv \sum_{\mathfrak{v} \in T(\lambda)} r_{\mathfrak{sv}}(a) c^\lambda_{\mathfrak{vt}} \mod A^{>\lambda}, \tag{3.20}$$
where $A^{>\lambda}$ is the subspace of A spanned by $\{c^\mu_{\mathfrak{ab}} \mid \mu > \lambda,\ \mathfrak{a}, \mathfrak{b} \in T(\mu)\}$.
(d) *$\deg : \amalg_{\lambda \in \Lambda} T(\lambda) \to \mathbb{Z}$ is a function and the elements of $\mathcal{C}$ are homogeneous with $\deg(c^\lambda_{\mathfrak{st}}) = \deg(\mathfrak{s}) + \deg(\mathfrak{t})$.*

The set $\mathcal{C}$ is called a graded cellular basis *of A.*

Let A be a graded cellular algebra. For $\lambda \in \Lambda$ let Δ^λ_k be the $\mathbb{F}_p$-linear span of some formal symbols $\{c^\lambda_{\mathfrak{s}} \mid \mathfrak{s} \in T(\lambda) \text{ and } \deg(\mathfrak{s}) = k\}$. Consider the direct sum
$$\Delta^\lambda := \bigoplus_{k \in \mathbb{Z}} \Delta^\lambda_k. \tag{3.21}$$

One defines an action of A on Δ^λ by
$$ac^\lambda_{\mathfrak{s}} = \sum_{\mathfrak{v} \in T(\lambda)} r_{\mathfrak{sv}}(a) c^\lambda_{\mathfrak{v}}, \tag{3.22}$$

where $r_{\mathfrak{sv}}(a)$ are the scalars that appear in Definition 3.3(c). We call the graded left A-module Δ^λ a *graded cell module*, where of course Δ^λ_k lives in degree k.

Given a $\mathbb{Z}$-graded A-module $M = \oplus_{k \in \mathbb{Z}} M_k$ we define its *graded dimension* as
$$\dim_v M = \sum_{k \in \mathbb{Z}} (\dim M_k)\, v^k \in \mathbb{Z}[v, v^{-1}]. \tag{3.23}$$

It is clear from the definitions that
$$\dim_v \Delta^\lambda = \sum_{\mathfrak{s} \in T(\lambda)} v^{\deg(\mathfrak{s})}. \tag{3.24}$$

Each graded cell module Δ^λ comes equipped with a symmetric and associative bilinear form $\langle\ ,\ \rangle_\lambda$. Such a form is determined by

$$c^\lambda_{\mathfrak{a}\mathfrak{s}} c^\lambda_{\mathfrak{t}\mathfrak{b}} \equiv \langle c^\lambda_{\mathfrak{s}}, c^\lambda_{\mathfrak{t}} \rangle_\lambda c^\lambda_{\mathfrak{a}\mathfrak{b}} \mod A^{>\lambda}. \tag{3.25}$$

The radical of this form, $\mathrm{Rad}(\Delta^\lambda) := \{x \in \Delta^\lambda \mid \langle x, y\rangle_\lambda = 0, \text{ for all } y \in \Delta^\lambda\}$, is a graded A-submodule of Δ^λ. Given $\lambda \in \Lambda$, we define $L^\lambda := \Delta^\lambda/\mathrm{Rad}(\Delta^\lambda)$ and

$$\Lambda_0 = \{\lambda \in \Lambda \mid L^\lambda \neq 0\}.$$

Theorem 3.4. [GL96, Theorem 3.4] [HM10, Theorem 2.10] *The set* $\{L^\lambda(j) \mid \lambda \in \Lambda_0$ *and* $j \in \mathbb{Z}\}$ *is a complete set of pairwise non-isomorphic graded simple* A*-modules.*

Let $\mathcal{G}(A)$ be the graded Grothendieck group of A. This is the $\mathcal{L}$-module generated by the symbols $[M]$, where M runs over the set of all finite-dimensional graded right A-modules, subject to the following relations:

- $[M(j)] = v^j[M]$, for all $j \in \mathbb{Z}$.
- $[M] = [N] + [P]$, if $0 \to N \to M \to P \to 0$ is a short exact sequence of graded right A-modules.

Corollary 3.5. *Let* A *be a graded cellular algebra. Then,* $\{[L^\lambda] \mid \lambda \in \Lambda_0\}$ *is an* $\mathcal{L}$*-basis of* $\mathcal{G}(A)$*.*

For any $\lambda \in \Lambda$ and $\mu \in \Lambda_0$, we define $d_{\lambda,\mu}(v) \in \mathcal{L}$, the *graded decomposition numbers of* A, by the formula

$$[\Delta^\lambda] = \sum_{\mu \in \Lambda_0} d_{\lambda,\mu}(v)[L^\mu]. \tag{3.26}$$

By general theory of graded cellular algebras [HM10, Lemma 2.13] we know that

$$d_{\mu,\mu}(v) = 1 \text{ and that } d_{\lambda,\mu}(v) = 0 \text{ if } \lambda \not\geq \mu, \tag{3.27}$$

for all $\lambda \in \Lambda$ and $\mu \in \Lambda_0$.

Definition 3.6. *Let* A *be a graded algebra and* M *be a graded* A*-module. We say that* M *is* positively graded *if* $\dim_v M \in \mathbb{Z}_{\geq 0}[v]$*. In addition, we say that* M *is* pure of degree d *if* $\dim_v M = nv^d$*, for some* $n \in \mathbb{N}$*. A* positively graded cellular algebra *is one in which the image of the function* deg *is in* $\mathbb{Z}_{\geq 0}$*.*

The following is a recollection of some useful facts about these kind of algebras.

Lemma 3.7. [HM10, Lemma 2.20] *Let* A *be a positively graded cellular algebra with graded cell datum as in Definition 3.3. Then,*

1. Δ^λ *is positively graded, for all* $\lambda \in \Lambda$*.*
2. *If* $\mu \in \Lambda_0$ *then there exists* $\mathfrak{t} \in T(\mu)$ *such that* $\deg(\mathfrak{t}) = 0$*.*
3. L^μ *is pure of degree zero, for all* $\mu \in \Lambda_0$*.*
4. $d_{\lambda,\mu}(v) \in \mathbb{Z}_{\geq 0}[v]$*, for all* $\lambda \in \Lambda$ *and* $\mu \in \Lambda_0$*.*

3.4. **A graded cellular basis for** $B_n(\kappa)$**.** In this section we recall the construction by Bowman [Bow17] of a graded cellular basis of $B^p_{l,n}(\kappa)$. His construction is for a larger family of algebras (see Remark 3.2). We omit the subtleties of this more general construction that do not arise in our context. In order to define the graded cellular basis for $B^p_{l,n}(\kappa)$ we need to introduce some combinatorial objects.

3.4.1. *One column multipartitions.* For $n \geq 0$, a *partition* λ of n is a weakly decreasing sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$. An *$l$-multipartition* $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ of n, is an l-tuple of partitions such that $|\lambda^{(1)}| + |\lambda^{(2)}| + \cdots |\lambda^{(l)}| = n$. The set of all l-multipartitions of n is denoted by $P^l(n)$.

In this paper, we are exclusively interested in *one-column l-multipartitions of* n, that is, l-multipartitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)})$ such that $0 \leq \lambda_1^{(m)} \leq 1$, for all $1 \leq m \leq l$. The set of all one-column l-multipartitions of n is denoted by $P_1^l(n)$. Another way of looking at this is to say that an element of $P_1^l(n)$ is a sequence of l terms $(1^{a_1}, \cdots, 1^{a_l})$ where $a_1 + \cdots + a_l = n$ and $a_i \in \mathbb{Z}_{\geq 0}$. For example, $\boldsymbol{\lambda} = (1^3, 1^4, 1^0, 1^3, 1^1) \in P_1^5(11)$.

For an element $\boldsymbol{\lambda} = (1^{a_1}, \cdots, 1^{a_l}) \in P_1^l(n)$ the *Young diagram* $[\boldsymbol{\lambda}]$ of $\boldsymbol{\lambda}$ is the set of "boxes" or "nodes"

$$\bigcup_{1 \leq m \leq l} \{(r, m) \mid 1 \leq r \leq a_m\}. \tag{3.28}$$

We say that a node $(r, m) \in [\boldsymbol{\lambda}]$ is in the r^{th} row of the m^{th} component of $\boldsymbol{\lambda}$.

Let (r, m) and (r', m') be two boxes. We say that (r, m) *dominates* (r', m') and write $(r, m) \rhd (r', m')$ if (r, m) is smaller than (r', m') in the lexicographical order, i.e., if $r < r'$ or $r = r'$ and $m < m'$. In the Young diagram, a box B is bigger than a box B' if B' is in the same row than B but strictly to the right or in a strictly lower row. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P_1^l(n)$. We say that $\boldsymbol{\lambda}$ *dominates* $\boldsymbol{\mu}$ and write $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ if for each box $(r, m) \in [\boldsymbol{\mu}]$ the number of boxes in $[\boldsymbol{\lambda}]$ that dominate (r, m) is greater or equal than the number of boxes in $[\boldsymbol{\mu}]$ that dominate (r, m).

Let $\boldsymbol{\lambda} \in P_1^l(n)$. A *tableau of shape* $\boldsymbol{\lambda}$ is a bijection $\mathfrak{t} : [\boldsymbol{\lambda}] \to \{1, \ldots, n\}$. We think of $\mathfrak{t}$ as filling in the boxes of $[\boldsymbol{\lambda}]$ with the numbers $1, 2, \ldots, n$. For $1 \leq i \leq n$, $\mathfrak{t}^{-1}(i)$ is the box of the Young diagram $[\boldsymbol{\lambda}]$ filled in by the number i.

A tableau is called *standard* if its entries increase from top to bottom on each component. We denote by $\operatorname{Std}(\boldsymbol{\lambda})$ the set of all standard tableaux of shape $\boldsymbol{\lambda}$. Given $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $1 \leq k \leq n$ we define $\mathfrak{t} \downarrow_k$ as the standard tableau obtained from $\mathfrak{t}$ by erasing the boxes with entries strictly greater than k in the order $\rhd$.

Let $\boldsymbol{\lambda} \in P_1^l(n)$. We say that a box $(r, m) \in [\boldsymbol{\lambda}]$ is *removable* from $\boldsymbol{\lambda}$ if $[\boldsymbol{\lambda}]/\{(r, m)\}$ is the Young diagram of some element in $P_1^l(n-1)$. Similarly, we say that a box $(r, m) \notin [\boldsymbol{\lambda}]$ is *addable* to $\boldsymbol{\lambda}$ if $[\boldsymbol{\lambda}] \cup \{(r, m)\}$ is the Young diagram of some element in $P_1^l(n+1)$. We denote by $\mathcal{R}(\boldsymbol{\lambda})$ (resp. $\mathcal{A}(\boldsymbol{\lambda})$) the set of all removable (resp. addable) boxes of $\boldsymbol{\lambda}$.

Let us define $\operatorname{res}(r, m)$ the *residue of the box* (r, m) by

$$\operatorname{res}(r, m) = \kappa_m + 1 - r \in I. \tag{3.29}$$

3.4.2. *Definition of* deg *in* $B_n(\kappa)$. This definition is not needed for the rest of the paper. We give it for completeness.

Given $1 \leq k \leq n$ and a standard tableaux $\mathfrak{t}$ we let $\mathcal{R}_{\mathfrak{t}}(k)$ (resp. $\mathcal{A}_{\mathfrak{t}}(k)$) denote the set of all boxes (r, m) satisfying the following three conditions:

(1) $(r, m) \in \mathcal{R}(\operatorname{Shape}(\mathfrak{t} \downarrow_k))$ (resp. $\mathcal{A}(\operatorname{Shape}(\mathfrak{t} \downarrow_k)))$
(2) $\operatorname{res}(r, m) = \operatorname{res}(\mathfrak{t}^{-1}(k))$
(3) $\mathfrak{t}^{-1}(k) \rhd (r, m)$

Let $\boldsymbol{\lambda} \in P_1^l(n)$ and $\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. The *degree* of $\mathfrak{t}$ is defined as

$$\deg(\mathfrak{t}) = \sum_{k=1}^{n} \left(|\mathcal{A}_{\mathfrak{t}}(k)| - |\mathcal{R}_{\mathfrak{t}}(k)|\right). \tag{3.30}$$

3.4.3. *Residue sequences and* $\mathfrak{t} \rightsquigarrow d_{\mathfrak{t}}$. For each $\boldsymbol{\lambda} \in P_1^l(n)$ we define a tableau $\mathfrak{t}^{\boldsymbol{\lambda}} \in \operatorname{Std}(\boldsymbol{\lambda})$ called the *dominant tableau* (for example, see the first two tableaux in Picture 3.34). It is determined by the following rule:

$$(\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(k) \rhd (\mathfrak{t}^{\boldsymbol{\lambda}})^{-1}(j) \text{ if and only if } k < j. \tag{3.31}$$

Given $\boldsymbol{\lambda} \in P_1^l(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ we denote by $d_{\mathfrak{t}} \in \mathfrak{S}_n$ the permutation determined by $d_{\mathfrak{t}}\mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$.

For each $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we define its *residue sequence* (see definition 3.29) as follows

$$\boldsymbol{i}^{\mathfrak{t}} = (\mathrm{res}(\mathfrak{t}^{-1}(1)), \mathrm{res}(\mathfrak{t}^{-1}(2)), \ldots, \mathrm{res}(\mathfrak{t}^{-1}(n))) \in I^n. \tag{3.32}$$

For notational convenience, we write $\boldsymbol{i}^{\boldsymbol{\lambda}} := \boldsymbol{i}^{\mathfrak{t}^{\boldsymbol{\lambda}}}$.

Definition 3.8. *Let $\boldsymbol{\lambda} \in P_1^l(n)$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. We fix reduced expressions $d_{\mathfrak{s}} = s_{i_1} \cdots s_{i_a}$ and $d_{\mathfrak{t}} = s_{j_1} \cdots s_{j_b}$. Then, we define*

$$\psi_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}} := \psi_{i_1} \cdots \psi_{i_a} e(\boldsymbol{i}^{\boldsymbol{\lambda}}) \psi_{j_b} \cdots \psi_{j_1} \in B_{l,n}^p(\kappa). \tag{3.33}$$

3.4.4. *The graded cellular basis.*

Proposition 3.9. [Bow17, Theorem 7.2] *The set $\{\psi_{\mathfrak{s}\mathfrak{t}}^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in P_1^l(n),\ \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})\}$ is a graded cellular basis of the generalized blob algebra $B_{l,n}^p(\kappa)$ with respect to the dominance order on $P_1^l(n)$, the degree function defined above and the involution $*$ determined by flipping upside down the diagrams.*

Once we have specified a graded cellular structure on $B_{l,n}^p(\kappa)$ we have automatically defined graded cell modules and graded simple modules. Given $\boldsymbol{\lambda} \in P_1^l(n)$, we will denote them by $\Delta^p(\boldsymbol{\lambda})$ and $L^p(\boldsymbol{\lambda})$, respectively. Since $\psi_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}} = e(\boldsymbol{\lambda})$ one can see that the bilinear form defined by the cellular structure on $\Delta^p(\boldsymbol{\lambda})$ is distinct from zero for each $\boldsymbol{\lambda} \in P_1^l(n)$. In other words, $L^p(\boldsymbol{\lambda}) \neq 0$, for each $\boldsymbol{\lambda} \in P_1^l(n)$. Then, we have well-defined graded decomposition numbers $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}^p \in \mathbb{Z}[v, v^{-1}]$, for each $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P_1^l(n)$.

Example 3.10. Let $n = 23$, $l = 4$, $e = 8$ and $\kappa = (0, 2, 4, 6) \in I_8^4$. Let $\boldsymbol{\lambda} = (1^1, 1^{13}, 1^1, 1^8)$, $\boldsymbol{\mu} = (1^5, 1^5, 1^6, 1^7) \in P_1^4(23)$. In (3.34) we have drawn the dominant tableaux $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}^{\boldsymbol{\mu}}$ corresponding to $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. We have also drawn another tableau $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\mu})$.

$$\mathfrak{t}^{\boldsymbol{\lambda}} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ & 5 & & 6 \\ & 7 & & 8 \\ & 9 & & 10 \\ & 11 & & 12 \\ & 13 & & 14 \\ & 15 & & 16 \\ & 17 & & 18 \\ & 19 & & \\ & 20 & & \\ & 21 & & \\ & 22 & & \\ & 23 & & \end{array} \qquad \mathfrak{t}^{\boldsymbol{\mu}} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ & & 21 & 22 \\ & & & 23 \end{array} \qquad \mathfrak{t} = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 9 & 5 & 10 & 6 \\ 11 & 7 & 12 & 8 \\ 13 & 18 & 14 & 17 \\ 15 & 23 & 16 & 19 \\ & & 22 & 20 \\ & & & 21 \end{array} \tag{3.34}$$

The permutation $d_{\mathfrak{t}} \in \mathfrak{S}_{23}$ is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 \\ 1 & 2 & 3 & 4 & 9 & 5 & 10 & 6 & 11 & 7 & 12 & 8 & 13 & 18 & 14 & 17 & 15 & 23 & 16 & 19 & 22 & 20 & 21 \end{pmatrix}.$$

Of course, there are many reduced expressions for $d_{\mathfrak{t}}$. However, any fixed but arbitrary choice of such a expression can be used to construct a graded cellular basis for $B^p_{l,n}(\kappa)$.

It is not hard to see that for any one-column l-multipartition $\boldsymbol{\lambda}$ the sets $\mathcal{A}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k)$ and $\mathcal{R}_{\mathfrak{t}^{\boldsymbol{\lambda}}}(k)$ are empty, for all $1 \leq k \leq n$. Therefore, we have $\deg(\mathfrak{t}^{\boldsymbol{\lambda}}) = \deg(\mathfrak{t}^{\boldsymbol{\mu}}) = 0$. On the other hand, for the tableau $\mathfrak{t}$ all the sets $\mathcal{R}_{\mathfrak{t}}(k)$ are empty, but there are six non-empty sets $\mathcal{A}_{\mathfrak{t}}(k)$, namely,

$$\begin{array}{ccc} \mathcal{A}_{\mathfrak{t}}(9) = \{(4,2)\}, & \mathcal{A}_{\mathfrak{t}}(10) = \{(4,4)\}, & \mathcal{A}_{\mathfrak{t}}(17) = \{(6,1)\}, \\ \mathcal{A}_{\mathfrak{t}}(18) = \{(6,3)\}, & \mathcal{A}_{\mathfrak{t}}(22) = \{(8,4)\}, & \mathcal{A}_{\mathfrak{t}}(23) = \{(7,3)\}. \end{array}$$

Therefore, $\deg(\mathfrak{t}) = 6$.

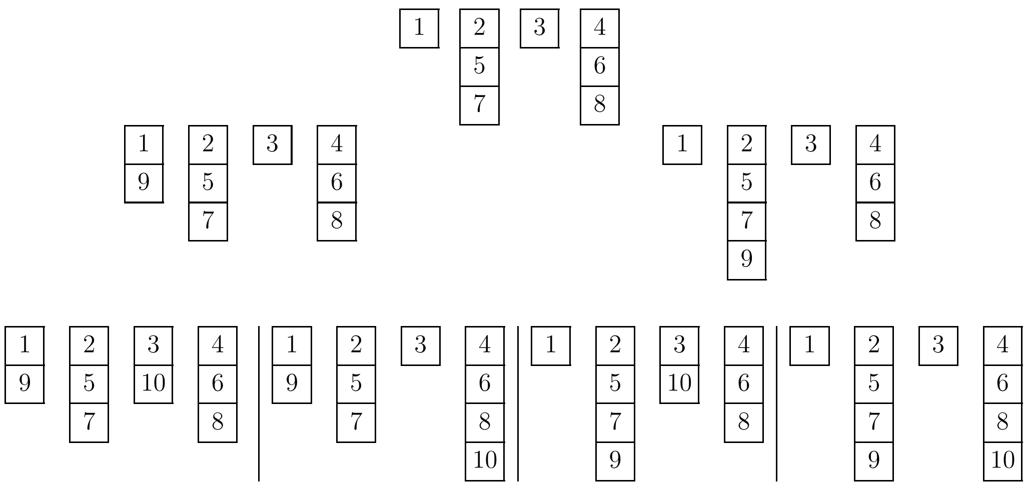
The residue sequences for the tableaux $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}^{\boldsymbol{\mu}}$ are given by

$$\boldsymbol{i}^{\boldsymbol{\lambda}} = (0,2,4,6,1,5,0,4,7,3,6,2,5,1,4,0,3,7,2,1,0,7,6),$$

$$\boldsymbol{i}^{\boldsymbol{\mu}} = (0,2,4,6,7,1,3,5,6,0,2,4,5,7,1,3,4,6,0,2,7,1,0).$$

We remark that the residue sequence associated to $\mathfrak{t}$ coincides with the one associated to $\mathfrak{t}^{\boldsymbol{\lambda}}$, that is, $\boldsymbol{i}^{\boldsymbol{\lambda}} = \boldsymbol{i}^{\mathfrak{t}}$. A main part of the forthcoming Section 4 is devoted to understand the set of all standard tableaux with the same residue sequence as some dominant tableau. For the moment, we want to roughly explain why are there 2^6 standard tableaux with residue sequence $\boldsymbol{i}^{\boldsymbol{\lambda}}$.

Let $\mathfrak{s}$ be a standard tableau with $\boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$. The first eight entries in $\mathfrak{s}$ are determined by $\boldsymbol{i}^{\boldsymbol{\lambda}}$, so that $\mathfrak{s}\downarrow_8$ corresponds to the tableau on top of the Figure (3.35). In Shape($\mathfrak{s}\downarrow_8$) we have two addable boxes with residue 7 (which is the ninth entry in $\boldsymbol{i}^{\boldsymbol{\lambda}}$), namely, the boxes $(2,1)$ and $(4,2)$. So that we can locate the number 9 in any of these two boxes keeping the same residue sequence as $\boldsymbol{i}^{\boldsymbol{\lambda}}$. Then we have two options for $\mathfrak{s}\downarrow_9$ which are drawn in the middle of (3.35). A similar situation occurs for the number 10, thus we have four options for $\mathfrak{s}\downarrow_{10}$, which are depicted in the bottom of (3.35). For any of the four options for $\mathfrak{s}\downarrow_{10}$ we can see that the number 11 is forced to be located below the number 9. This is because for any of these tableaux the box below the number 9 is the unique addable box with residue 6 (which is the eleventh entry in $\boldsymbol{i}^{\boldsymbol{\lambda}}$). Then, we still have only four options for $\mathfrak{s}\downarrow_{11}$. By continuing in this way we can see that in each step we will have one or two options to locate the next numbers. The numbers in which we will have two possibilities are 17, 18, 22 and 23. Therefore, we have 2^6 standard tableaux $\mathfrak{s}$ with $\boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$.



(3.35)

3.5. **Idempotent truncations of $B^p_{l,n}(\kappa)$.** In this section we consider certain subalgebras of $B^p_{l,n}(\kappa)$ which are obtained by idempotent truncation. They will be closely related to the light leaves basis. As before, $e, l, n > 0$ and κ is an increasing adjacency-free multicharge.

We define the *idempotent truncation of* $B^p_{l,n}(\kappa)$ *at* $\boldsymbol{\lambda}$ as $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa) = e(\boldsymbol{i}^{\boldsymbol{\lambda}})B^p_{l,n}(\kappa)e(\boldsymbol{i}^{\boldsymbol{\lambda}})$, for each $\boldsymbol{\lambda} \in P^l_1(n)$. It follows from the relations (3.2), (3.5) and from the definition 3.8 that

$$e(\boldsymbol{i}^{\boldsymbol{\lambda}})\psi^{\boldsymbol{\mu}}_{\mathfrak{st}}e(\boldsymbol{i}^{\boldsymbol{\lambda}}) = \begin{cases} \psi^{\boldsymbol{\mu}}_{\mathfrak{st}}, & \text{if } \boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.36}$$

Therefore, $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$ has a basis which is a subset of the graded cellular basis of $B^p_{l,n}(\kappa)$. Let us be more precise. For any $\boldsymbol{\mu} \in P^l_1(n)$ we define $\operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \{\mathfrak{t} \in \operatorname{Std}(\boldsymbol{\mu}) \mid \boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}\}$. We also define $P(\boldsymbol{\lambda}) = \{\boldsymbol{\mu} \in P^l_1(n) \mid \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \neq \emptyset\}$.

Lemma 3.11. *For each* $\boldsymbol{\lambda} \in P^l_1(n)$ *the set* $\{\psi^{\boldsymbol{\mu}}_{\mathfrak{st}} \mid \boldsymbol{\mu} \in P(\boldsymbol{\lambda}),\, \mathfrak{s}, \mathfrak{t} \in \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})\}$ *is a graded cellular basis of* $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$ *with respect to the order, degree and involution inherited of* $B^p_{l,n}(\kappa)$.

Proof: It follows directly from (3.36). □

Given $\boldsymbol{\mu} \in P(\boldsymbol{\lambda})$, we denote by $\Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ and $L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ the corresponding graded cell and simple modules of $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$, respectively. It is not hard to see that $\Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \simeq e(\boldsymbol{i}^{\boldsymbol{\lambda}})\Delta^p(\boldsymbol{\mu})$ and $L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \simeq e(\boldsymbol{i}^{\boldsymbol{\lambda}})L^p(\boldsymbol{\mu})$. As $L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is a quotient of $\Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$,

$$\Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = 0 \Rightarrow L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = 0. \tag{3.37}$$

The following formula is Equation 3.24 applied to this set up.

$$\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \sum_{\mathfrak{t} \in \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})} v^{\deg(\mathfrak{t})}. \tag{3.38}$$

Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in P(\boldsymbol{\lambda})$, with $L_{\boldsymbol{\lambda}}(\boldsymbol{\nu}) \neq 0$. We denote by $d^{p,\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\nu}}$ the graded decomposition number associated to $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ in $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$. Graded decomposition numbers of $B^p_{l,n}(\kappa)$ and $B^{p,\boldsymbol{\lambda}}_{l,n}(\kappa)$ are compatible, this is, $d^p_{\boldsymbol{\mu},\boldsymbol{\nu}} = d^{p,\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\nu}}$ if $d^{p,\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\nu}}$ is defined (see for instance the Appendix in [Don98]). On the other hand, suppose we want to calculate $d^p_{\boldsymbol{\mu},\boldsymbol{\nu}}$, for some pair $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in P^l_1(n)^2$. In this case we work with $B^{p,\boldsymbol{\nu}}_{l,n}(\kappa)$. We have $\operatorname{Std}_{\boldsymbol{\nu}}(\boldsymbol{\nu}) = \{\mathfrak{t}^{\boldsymbol{\nu}}\}$. Therefore,

$$\Delta_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \cong L_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \cong \operatorname{Span}_{\mathbb{F}_p}\{\psi_{\mathfrak{t}^{\boldsymbol{\nu}}}\}. \tag{3.39}$$

In particular, $L_{\boldsymbol{\nu}}(\boldsymbol{\nu}) \neq 0$. If $\boldsymbol{\mu} \in P(\boldsymbol{\nu})$ then $d^{p,\boldsymbol{\nu}}_{\boldsymbol{\mu},\boldsymbol{\nu}}$ is defined, so that $d^{p,\boldsymbol{\nu}}_{\boldsymbol{\mu},\boldsymbol{\nu}} = d^p_{\boldsymbol{\mu},\boldsymbol{\nu}}$. If $\boldsymbol{\mu} \notin P(\boldsymbol{\nu})$ (and therefore $e(\boldsymbol{i}^{\boldsymbol{\nu}})\Delta^p(\boldsymbol{\mu}) = 0$) then we can again use the results in [Don98, Appendix] to obtain $d^p_{\boldsymbol{\mu},\boldsymbol{\nu}} = 0$. Summing up, we conclude that each non-zero graded decomposition number of $B^p_{l,n}(\kappa)$ arises as a graded decomposition number of some idempotent truncation. For this reason, in the forthcoming sections we focus on study idempotent truncations of $B^p_{l,n}(\kappa)$ rather than $B^p_{l,n}(\kappa)$ itself.

4. Alcove geometry

The main goal of this section is to prove that $\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ coincides with $\deg(\mathbb{L}_{\underline{w_{\boldsymbol{\lambda}}}}(w_{\boldsymbol{\mu}}))$, where $w_{\boldsymbol{\lambda}}$ and $w_{\boldsymbol{\mu}}$ are some elements in W_l that we will introduce and $\underline{w_{\boldsymbol{\lambda}}}$ is a particular reduced expression of $w_{\boldsymbol{\lambda}}$. To do this it is convenient to interpret standard one-column tableaux as paths in an alcove geometry.

4.1. **Affine reflections and alcoves.** Let E_l denote the quotient space $\mathbb{R}^l/\langle \epsilon_1 + \epsilon_2 + \cdots + \epsilon_l \rangle$, where ϵ_i denotes the i-th coordinate vector of $\mathbb{R}^l$. Let $\odot$ denote the origin in E_l. We identify elements $\boldsymbol{\lambda} \in P^l_1(n)$ with points (resp. classes) in $\mathbb{R}^l$ (resp. E_l) via the map $\boldsymbol{\lambda} = (1^{a_1}, \ldots 1^{a_l}) \mapsto \sum_{i=1}^l a_i\epsilon_i$. For each $1 \leq i < j \leq l$ and each $m \in \mathbb{Z}$ define the affine hyperplane

$$\mathfrak{h}^m_{i,j} = \{x \in E_l \mid x_i - x_j = \kappa_i - \kappa_j + me\} \tag{4.1}$$

and $s^m_{i,j}$ the corresponding affine reflection. In formulas

$$s^m_{i,j}(x) = x - ((x_i - x_j) - (\kappa_i - \kappa_j + me))(\epsilon_i - \epsilon_j). \tag{4.2}$$

Let W_l be the affine Weyl group of type $\tilde{A}_{l-1}$. We identify W_l with the group generated by the affine reflections via the assignment $s_0 \mapsto s^1_{1,l}$ and $s_i \mapsto s^0_{i,i+1}$, for $1 \leq i < l$.

Let $\mathfrak{H}$ be the union of all the affine hyperplanes.The connected components of $E_l \backslash \mathfrak{H}$ are called *alcoves*. Let $\mathcal{A}$ denote the set of alcoves. A point $x \in E_l$ is called *regular* if it belongs to some alcove and $\boldsymbol{\lambda} \in P^l_1(n)$ is called *regular* if its image in E_l is regular. By the restriction imposed on κ it is clear that $\odot$ is regular.

The alcove containing $\odot$ is called the fundamental alcove and is denoted by A_0. It is well-known that A_0 has l walls which are supported by the hyperplanes $\{\mathfrak{h}^0_{i,i+1} \mid 1 \leq i < l\}$ and $\mathfrak{h}^1_{1,l}$. The action of W_l on E_l extends to an action on $\mathcal{A}$. Given $A \in \mathcal{A}$ there exists a unique $w \in W_l$ such that $w \cdot A_0 = A$. If this is the case, we denote $A := A_w$.

4.2. **Paths and sequences of hyperplanes associated to standard tableaux.**

Definition 4.1. *Given $\boldsymbol{\lambda} \in P^l_1(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ we define the path associated to $\mathfrak{t}$ as the piecewise linear path $p_\mathfrak{t} : [0, n] \to E_l$ with vertices $p_\mathfrak{t}(0) = \odot$ and $p_\mathfrak{t}(k) = Shape(\mathfrak{t} \downarrow_k) \in E_l$, for $1 \leq k \leq n$. We denote $p_{\boldsymbol{\lambda}} := p_{\mathfrak{t}^{\boldsymbol{\lambda}}}$.*

We remark that if $\boldsymbol{\lambda} \in P^l_1(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ then $p_\mathfrak{t}(n) = \boldsymbol{\lambda}$. In particular, $p_{\boldsymbol{\lambda}}(n) = \boldsymbol{\lambda}$. The following lemma is [BCS17, Lemma 4.7].

Lemma 4.2. *Let $\boldsymbol{\lambda} \in P^l_1(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Then, the vertex $p_\mathfrak{t}(k)$ of the path $p_\mathfrak{t}$ is in the hyperplane $\mathfrak{h}^m_{i,j}$ if and only if the addable boxes in the i-th and j-th components of $Shape(\mathfrak{t} \downarrow_k)$ have the same residue.*

From now and on we fix $\boldsymbol{\lambda} \in P^l_1(n)$ regular. In accordance with Lemma 4.2, the regularity of $\boldsymbol{\lambda}$ is equivalent to the fact that the boxes addable to $\boldsymbol{\lambda}$ have different residues. We denote by $w_{\boldsymbol{\lambda}} \in W_l$ the unique element such that $\boldsymbol{\lambda} \in A_{w_{\boldsymbol{\lambda}}}$. In order to establish a connection between this alcove geometry and the light leaves basis we need a particular reduced expression for $w_{\boldsymbol{\lambda}}$. To do this we begin by associating to $\boldsymbol{\lambda}$ a sequence of hyperplanes $\mathfrak{h}_{\boldsymbol{\lambda}}$ given, in order, by the new hyperplanes touched by the path $p_{\boldsymbol{\lambda}}$. Let us be more precise. For $1 \leq k \leq n$ we define

$$\mathfrak{h}(k, \boldsymbol{\lambda}) = \{\mathfrak{h} \in \mathfrak{H} \mid p_{\boldsymbol{\lambda}}(k) \in \mathfrak{h}\} \backslash \{\mathfrak{h} \in \mathfrak{H} \mid p_{\boldsymbol{\lambda}}(k-1) \in \mathfrak{h}\}. \tag{4.3}$$

We want to prove that the set $\mathfrak{h}(k, \boldsymbol{\lambda})$ consists of at most one element, for all $1 \leq k \leq n$. We will first need a preliminary result.

Lemma 4.3. *Let $\boldsymbol{\lambda} \in P^l_1(n)$ regular. Then, $Shape(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_k)$ has at most two addable boxes with the same residue, for each $0 \leq k \leq n$.*

Proof: Suppose there exists k such that $\mathrm{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_k)$ has three addable boxes A_1, A_2 and A_3 with the same residue. Let r_i be the row to which A_i belongs and r the row to which the box occupied by k in $\mathfrak{t}^{\boldsymbol{\lambda}}$ belongs. Since κ is assumed to be adjacency-free we know that r_1, r_2 and r_3 are different and not adjacent, that is, $r_i \neq r_j \pm 1$. In particular, two of the elements of $\{r_1, r_2, r_3\}$ are less than $r - 1$. By definition of $\mathfrak{t}^{\boldsymbol{\lambda}}$, the numbers greater than k are placed in rows lower than or equal to the r^{th} row. It follows that at least two of the boxes A_1, A_2 and A_3 are addable to $\mathrm{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_{k'})$, for all $k \leq k' \leq n$. In particular, we have that $\boldsymbol{\lambda} = \mathrm{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_n)$ has two addable boxes with the same residue, which contradicts the hypothesis of $\boldsymbol{\lambda}$ being regular. □

Corollary 4.4. *Let $\boldsymbol{\lambda} \in P^l_1(n)$ be regular. Then, the set $\mathfrak{h}(k, \boldsymbol{\lambda})$ consists of at most one element, for all $1 \leq k \leq n$.*

Proof: Let $1 \leq k \leq n$. Let A be the box occupied by k in $\mathfrak{t}^{\boldsymbol{\lambda}}$ and let i be the component to which A belongs. The paths $p_{\boldsymbol{\lambda}}(k-1)$ and $p_{\boldsymbol{\lambda}}(k)$ only differ in the i-th component. So that the unique addable box to $\mathrm{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_k)$ which is not addable to $\mathrm{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_{k-1})$ is the one located just below A.

If $\mathfrak{h}(k, \boldsymbol{\lambda})$ has two or more elements, then Lemma 4.2 implies Shape($\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_k$) has three or more addable boxes with the same residue, contradicting Lemma 4.3. $\square$

Let $k_1 < k_2 < \ldots < k_r$ be the integers such that $\mathfrak{h}(k_i, \boldsymbol{\lambda}) \neq \emptyset$. In accordance with Corollary 4.4, we define $\mathfrak{h}_i$ to be the unique hyperplane belonging to $\mathfrak{h}(k_i, \boldsymbol{\lambda})$. Then, we define the *sequence of hyperplanes associated to* $\boldsymbol{\lambda}$ by $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}_1, \ldots, \mathfrak{h}_r)$. By convention, if $\boldsymbol{\lambda} \in A_0$ then $\mathfrak{h}_{\boldsymbol{\lambda}}$ is the empty sequence.

It is possible that some vertex $p_{\boldsymbol{\lambda}}(k)$ of $p_{\boldsymbol{\lambda}}$ belongs to the intersection of two or more hyperplanes. For instance, in Example 3.10, $p_{\boldsymbol{\lambda}}(8)$ belongs to $\mathfrak{h}^0_{1,2} \cap \mathfrak{h}^0_{3,4}$. However, if some vertex of $p_{\boldsymbol{\lambda}}$ is on the intersection of some hyperplanes, say $\mathfrak{h}^{m_1}_{i_1,j_1}$ and $\mathfrak{h}^{m_2}_{i_2,j_2}$, then, because of Lemma 4.2 and Lemma 4.3, we have that $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$. In this case, following [BCS17, Definition 2.7], we say the hyperplanes $\mathfrak{h}^{m_1}_{i_1,j_1}$ and $\mathfrak{h}^{m_2}_{i_2,j_2}$ are *orthogonal.*

Example 4.5. We keep the same parameters as in Example 3.10, so that $\kappa = (0, 2, 4, 8)$, $e = 8$ and $\boldsymbol{\lambda} = (1^1, 1^{13}, 1^1, 1^8)$. We want to determine the integers k_i such that $\mathfrak{h}(k_i, \boldsymbol{\lambda}) \neq \emptyset$. According to Lemma 4.2, these integers will be the ones satisfying that Shape($\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_{k_i}$) has a pair of addable boxes of the same residue but one of the boxes in this pair is not addable to Shape($\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_{k_i-1}$). So, we have

$$k_1 = 7,\ k_2 = 8,\ k_3 = 15,\ k_4 = 16,\ k_5 = 21,\ k_6 = 22.$$

The sequence $\mathfrak{h}_{\boldsymbol{\lambda}}$ is given by $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}^0_{1,2}, \mathfrak{h}^0_{3,4}, \mathfrak{h}^1_{2,3}, \mathfrak{h}^0_{1,4}, \mathfrak{h}^{-1}_{1,2}, \mathfrak{h}^1_{2,4})$.

4.3. **Some properties of $\mathfrak{h}_{\boldsymbol{\lambda}}$.** Each hyperplane $\mathfrak{h}^m_{i,j} \in \mathfrak{H}$ splits E_l into two half-spaces

$$E^m_{i,j}(+) := \{x \in E_l \mid x_i - x_j > \kappa_i - \kappa_j + me\} \text{ and } E^m_{i,j}(-) := \{x \in E_l \mid x_i - x_j < \kappa_i - \kappa_j + me\}. \quad (4.4)$$

We say that a hyperplane *separates* two alcoves if they belong to distinct half-spaces. It is a known fact that $l(w)$ coincides with the number of hyperplanes separating A_w from A_0 (see, for example, [Hum92]).

Lemma 4.6. *Let $\boldsymbol{\lambda} = (1^{a_1}, \ldots, 1^{a_l}) \in P^l_1(n)$ regular and let $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}_1, \ldots \mathfrak{h}_r)$ be the corresponding sequence of hyperplanes. Then,*

1. *The hyperplanes $\mathfrak{h}_i$'s are all distinct.*
2. *Each $\mathfrak{h}_i$ separates $A_{w_{\boldsymbol{\lambda}}}$ and A_0.*
3. *Any hyperplane separating $A_{w_{\boldsymbol{\lambda}}}$ and A_0 belongs to $\mathfrak{h}_{\boldsymbol{\lambda}}$.*
4. *$l(w_{\boldsymbol{\lambda}}) = r$.*

Proof: Let us prove (1). Let us suppose that the hyperplane $\mathfrak{h}^m_{i,j}$ belongs to $\mathfrak{h}_{\boldsymbol{\lambda}}$. We must have $a_i \neq a_j$ because if $a_i = a_j$, by the way that the numbers are filled in $\mathfrak{t}^{\boldsymbol{\lambda}}$, for any k the addable boxes in the i-th and j-th components of Shape($\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_k$) would never have the same residue since κ is adjacency-free. This, by Lemma 4.2 contradicts the hypothesis that the hyperplane $\mathfrak{h}^m_{i,j}$ belongs to $\mathfrak{h}_{\boldsymbol{\lambda}}$.

Without loss of generality, assume $a_i > a_j$. For $1 \leq k \leq n$, we set

$$(1^{\mu^k_1}, \ldots, 1^{\mu^k_l}) := \text{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_k) = p_{\boldsymbol{\lambda}}(k). \quad (4.5)$$

Let a be the minimal integer satisfying $\mathfrak{h}(a, \boldsymbol{\lambda}) = \{\mathfrak{h}^m_{i,j}\}$ and let b be the integer which occurs in the box located just below the box occupied by a in $\mathfrak{t}^{\boldsymbol{\lambda}}$ (in the i-th component). This number must exist, otherwise $\boldsymbol{\lambda}$ would belong to $\mathfrak{h}^m_{i,j}$ and it would not be regular. Let us remark that since $a_i > a_j$ we have $a_j = \mu^k_j$ for all $a \leq k \leq n$.

- If $a < k < b$ then $p_{\boldsymbol{\lambda}}(k)$ and $p_{\boldsymbol{\lambda}}(k-1)$ belong to $\mathfrak{h}^m_{i,j}$, and therefore $\mathfrak{h}^m_{i,j} \notin \mathfrak{h}(k, \boldsymbol{\lambda})$.
- If $b \leq k \leq n$ then $\mu^k_i - \mu^k_j > \mu^a_i - \mu^a_j = \kappa_i - \kappa_j + me$, thus $p_{\boldsymbol{\lambda}}(k) \notin \mathfrak{h}^m_{i,j}$.

We conclude that $\mathfrak{h}^m_{i,j}$ cannot occur twice in $\mathfrak{h}_{\boldsymbol{\lambda}}$, thus proving (1).

Let us prove (2). Let $\mathfrak{h}^m_{i,j}$ be a hyperplane in $\mathfrak{h}_{\boldsymbol{\lambda}}$. Since κ is increasing and adjacency-free we know that $-e < \kappa_i - \kappa_j < 0$. Therefore, $\odot \in E^m_{i,j}(+)$ if $m < 1$ and $\odot \in E^m_{i,j}(-)$

if $m \geq 1$. Let a be the (unique) integer given by $\mathfrak{h}(a, \boldsymbol{\lambda}) = \{\mathfrak{h}_{i,j}^m\}$. As before, we set $(1^{\mu_1^a}, \ldots, 1^{\mu_l^a}) = \mathrm{Shape}(\mathfrak{t}^{\boldsymbol{\lambda}} \downarrow_a)$. We have

$$\mu_i^a - \mu_j^a = \kappa_i - \kappa_j + me. \tag{4.6}$$

Suppose that $m \geq 1$ (the case $m < 1$ is similar). By (4.6) we obtain $\mu_i^a > \mu_j^a$. As in point (1) $\mu_j^a = a_j$ and $\mu_i^a < a_i$. It follows that $a_i - a_j > \mu_i^a - \mu_j^a = \kappa_i - \kappa_j + me$. Thus, $\boldsymbol{\lambda} \in E_{i,j}^m(+)$ and $\mathfrak{h}_{i,j}^m$ separates $A_{w_{\boldsymbol{\lambda}}}$ and A_0. This proves (2).

If an hyperplane separates A_0 and $A_{w_{\boldsymbol{\lambda}}}$, it must be hit by $p_{\boldsymbol{\lambda}}$ (a continuous path) thus it belongs to $\mathfrak{h}_{\boldsymbol{\lambda}}$ and (3) follows. (4) is a direct consequence of the previous statements. □

4.4. **The alcove sequence $\mathfrak{a}_{\boldsymbol{\lambda}}$.** Let $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}_1, \cdots, \mathfrak{h}_r)$ be the sequence of hyperplanes associated to $\boldsymbol{\lambda}$. Let us define a sequence of alcoves $\mathfrak{a}_{\boldsymbol{\lambda}} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ as follows. Firstly define $\mathfrak{a}_0 = A_0$. The following alcoves in the sequence are inductively defined by $\mathfrak{a}_i = \rho_i \mathfrak{a}_{i-1}$, where ρ_i denotes the orthogonal reflection through the hyperplane $\mathfrak{h}_i$.
Let $k_1 < k_2 < \ldots < k_r$ be the integers such that $\mathfrak{h}(k_i, \boldsymbol{\lambda}) = \{\mathfrak{h}_i\}$. Set $p_0 = p_{\boldsymbol{\lambda}}$. We are going to define paths $p_1, p_2, \ldots, p_r$ as follows. Suppose the path p_i has been already defined. Then, we define p_{i+1} as the path obtained from p_i by applying the reflection $\rho_1 \rho_2 \cdots \rho_i \cdots \rho_2 \rho_1$ to all the vertices of p_i after the vertex $p_i(k_{i+1})$. For instance, keeping the same notation and parameters as in Example 3.10 the path p_r is the one which corresponds to the tableu $\mathfrak{t}$. Since $\boldsymbol{\lambda}$ is regular we know that $k_r < n$, so that in each step of the above construction the relevant reflection was applied to last vertex. In other words,

$$\begin{aligned} p_r(n) &= (\rho_1 \rho_2 \cdots \rho_r \cdots \rho_2 \rho_1) \cdots (\rho_1 \rho_2 \rho_1)(\rho_1)(p_0(n)) \\ &= (\rho_1 \rho_2 \cdots \rho_r \cdots \rho_2 \rho_1) \cdots (\rho_1 \rho_2 \rho_1)(\rho_1)(\boldsymbol{\lambda}) \\ &= \rho_1 \rho_2 \cdots \rho_r(\boldsymbol{\lambda}). \end{aligned}$$

Since $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}_1, \cdots, \mathfrak{h}_r)$ corresponds to the sequence of all hyperplanes touched by $p_{\boldsymbol{\lambda}} = p_0$ in order, the path p_r is completely contained in the closure of the fundamental alcove. In particular, $p_r(n)$ belongs to the closure of the fundamental alcove. As $\boldsymbol{\lambda}$ is regular and regularity is not affected by the action of W_l we conclude that $p_r(n) = \rho_1 \rho_2 \cdots \rho_r(\boldsymbol{\lambda})$ belongs to the fundamental alcove. Therefore, $w_{\boldsymbol{\lambda}} = \rho_r \cdots \rho_2 \rho_1$.
Given two alcoves A and B we say that they are *adjacent* if B is obtained from A by applying an affine reflection with respect to one of the walls of A. In other words, two alcoves are adjacent if they are different and share a common wall. We remark that two adjacent alcoves share $l-2$ common walls. A sequence of alcoves $(\mathfrak{b}_0, \mathfrak{b}_1, \ldots, \mathfrak{b}_r)$ is called an *alcove path* if $\mathfrak{b}_0 = A_0$ and $\mathfrak{b}_{i-1}$ and $\mathfrak{b}_i$ are adjacent, for $1 \leq i \leq r$. An alcove path $(\mathfrak{b}_0, \mathfrak{b}_1, \ldots, \mathfrak{b}_r)$ is called *reduced* if r is minimal among the set of all alcove paths ending in $\mathfrak{b}_r$.

Lemma 4.7. *Let $\boldsymbol{\lambda} \in P_1^l(n)$ regular. The alcove sequence $\mathfrak{a}_{\boldsymbol{\lambda}} = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_r)$ of $\boldsymbol{\lambda}$ is a reduced alcove path.*

Proof: Let $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}_1, \ldots, \mathfrak{h}_r)$ be the hyperplane sequence associated to $\boldsymbol{\lambda}$. Recall that ρ_i is the reflection with respect the hyperplane $\mathfrak{h}_i$. Since by definition $\mathfrak{a}_0 = A_0$, in order to prove that $\mathfrak{a}_{\boldsymbol{\lambda}}$ is an alcove path it is enough to show that $\mathfrak{h}_i$ is a wall of $\mathfrak{a}_{i-1}$. We proceed by induction. The hyperplane $\mathfrak{h}_1$ is the first hyperplane intersected by $p_{\boldsymbol{\lambda}}$. As $p_{\boldsymbol{\lambda}}$ is continuous and it begins at $\odot \in A_0$ we conclude that $\mathfrak{h}_1$ must be a wall of the fundamental alcove $A_0 = \mathfrak{a}_0$. This yields the base of our induction.
We now suppose that $i > 1$ and that $\mathfrak{h}_{i-1}$ is a wall of $\mathfrak{a}_{i-2}$. As $\mathfrak{a}_{i-1} = \rho_{i-1} \mathfrak{a}_{i-2}$ we know that $\mathfrak{h}_{i-1}$ is a wall of $\mathfrak{a}_{i-1}$ as well. Let k_{i-1} and k_i be the integers such that $\mathfrak{h}(k_{i-1}, \boldsymbol{\lambda}) = \{\mathfrak{h}_{i-1}\}$ and $\mathfrak{h}(k_i, \boldsymbol{\lambda}) = \{\mathfrak{h}_i\}$, respectively. Then, $p_{\boldsymbol{\lambda}}(k_{i-1}) \in \mathfrak{h}_{i-1}$ and $p_{\boldsymbol{\lambda}}(k_i) \in \mathfrak{h}_i$. The proof splits into two cases.

Case A: $p_{\boldsymbol{\lambda}}(k_i) \in \mathfrak{h}_{i-1}$**.** In this case we have $p_{\boldsymbol{\lambda}}(k_i) \in \mathfrak{h}_{i-1} \cap \mathfrak{h}_i$ and therefore $\mathfrak{h}_{i-1}$ and $\mathfrak{h}_i$ are orthogonal. Let $\mathfrak{h}$ be a wall of $\mathfrak{a}_{i-2}$ distinct of $\mathfrak{h}_{i-1}$ and orthogonal to $\mathfrak{h}_{i-1}$. Since $\mathfrak{a}_{i-1} = \rho_{i-1}\mathfrak{a}_{i-2}$ we obtain that $\mathfrak{h} = \rho_{i-1}\mathfrak{h}$ is a wall of $\mathfrak{a}_{i-1}$. So, it suffices to prove that $\mathfrak{h}_i$ is a wall of $\mathfrak{a}_{i-2}$. This follows by noticing that from the vertex k_{i-1} to the vertex k_i the path $p_{\boldsymbol{\lambda}}$ is always on $\mathfrak{h}_{i-1}$ and it does not intersect a new hyperplane till its intersection with $\mathfrak{h}_i$. Then, as $\mathfrak{h}_{i-1}$ is a wall of $\mathfrak{a}_{i-2}$, $\mathfrak{h}_i$ must also be a wall of $\mathfrak{a}_{i-2}$.

Case B: $p_{\boldsymbol{\lambda}}(k_i) \notin \mathfrak{h}_{i-1}$**.** In this case the path $p_{\boldsymbol{\lambda}}$ leaves the hyperplane $\mathfrak{h}_{i-1}$ before the vertex k_i. Actually, by the way the numbers are filled in $\mathfrak{t}^{\boldsymbol{\lambda}}$, $p_{\boldsymbol{\lambda}}$ must leave any hyperplane before the vertex k_i. In other words, $p_{\boldsymbol{\lambda}}(k_i - 1)$ belongs to an alcove. By construction this alcove must be $\mathfrak{a}_{i-1}$. Therefore, the next hyperplane that $p_{\boldsymbol{\lambda}}$ intersects must be a wall of $\mathfrak{a}_{i-1}$. But this hyperplane is by definition $\mathfrak{h}_i$.

By [LP07, Lemma 5.3] we know that the number of alcoves involved in a reduced alcove path ending in $A_{w_{\boldsymbol{\lambda}}}$ is $l(w_{\boldsymbol{\lambda}}) + 1$. Then, $\mathfrak{a}_{\boldsymbol{\lambda}}$ is reduced by Lemma 4.6. □

We are now in position to obtain a reduced expression for $w_{\boldsymbol{\lambda}}$. Recall that $\mathfrak{h}_{\boldsymbol{\lambda}} = (\mathfrak{h}_1, \dots, \mathfrak{h}_r)$ and $\mathfrak{a}_{\boldsymbol{\lambda}} = (\mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_r)$ are the hyperplane and alcove sequences associated to $\boldsymbol{\lambda}$. Recall that ρ_j is the reflection through $\mathfrak{h}_j$. By definition of $\mathfrak{a}_{\boldsymbol{\lambda}}$ we know that $\mathfrak{a}_j = \rho_j\rho_{j-1}\cdots\rho_1\mathfrak{a}_0$, for each $1 \leq j \leq r$. We remark that if A and B are adjacent alcoves and $w \in W_l$ then wA and wB are also adjacent. Then, applying $\rho_1\cdots\rho_{j-1}$ to the pair of adjacent alcoves $\mathfrak{a}_{j-1}$ and $\mathfrak{a}_j$, we obtain the pair of adjacent alcoves $\rho_1\cdots\rho_{j-1}\mathfrak{a}_{j-1} = \mathfrak{a}_0$ and $\rho_1\cdots\rho_{j-1}\mathfrak{a}_j = \rho_1\cdots\rho_{j-1}\rho_j\rho_{j-1}\cdots\rho_1\mathfrak{a}_0$. It follows that $\rho_1\cdots\rho_{j-1}\rho_j\rho_{j-1}\cdots\rho_1$ is a reflection with respect some wall of the fundamental alcove. So, for each $1 \leq j \leq r$, there is an $i_j \in \{0, 1, \dots, l-1\}$ satisfying that $\rho_1\cdots\rho_{j-1}\rho_j\rho_{j-1}\cdots\rho_1$ is the simple reflection s_{i_j}. Clearly, we have

$$s_{i_1}\cdots s_{j_r} = (\rho_1)(\rho_1\rho_2\rho_1)\cdots(\rho_1\rho_2\cdots\rho_{r-1}\rho_r\rho_{r-1}\cdots\rho_2\rho_1) = \rho_r\rho_{r-1}\cdots\rho_2\rho_1 = w_{\boldsymbol{\lambda}}.$$

Therefore, $\underline{w_{\boldsymbol{\lambda}}} = s_{i_1}\cdots s_{i_r}$ is a reduced expression of $w_{\boldsymbol{\lambda}}$ which we call the *principal reduced expression of* $w_{\boldsymbol{\lambda}}$.

Example 4.8. With the same notation and parameters as in Example 3.10 we have

$$\underline{w_{\boldsymbol{\lambda}}} = s_1s_3s_0s_2s_3s_2.$$

Theorem 4.9. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P_1^l(n)$. Suppose that $\boldsymbol{\lambda}$ is regular.*

1. *If $\boldsymbol{\mu}$ does not belong to the orbit of $\boldsymbol{\lambda}$ then* $\operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \emptyset$.
2. *If $\boldsymbol{\mu}$ belongs to the orbit of $\boldsymbol{\lambda}$ then $\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ coincides with the coefficient of $H_{w_{\boldsymbol{\mu}}}$ in the expansion of $H_{\underline{w_{\boldsymbol{\lambda}}}}$ in terms of the standard basis of the Hecke algebra of W_l. Consequently,*

$$\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \deg \mathbb{L}_{\underline{w_{\boldsymbol{\lambda}}}}(w_{\boldsymbol{\mu}}). \tag{4.7}$$

Proof: Let $k_1 < k_2 < \ldots < k_r$ be the integers such that $\mathfrak{h}(k_i, \boldsymbol{\lambda}) \neq \emptyset$. We construct a perfect binary tree with nodes decorated by paths starting at $\odot$. We construct it by induction on the level of depth. In depth zero the unique node is decorated by $p_{\boldsymbol{\lambda}}$. Suppose now a node of depth s has been decorated by a path p. One of the two child nodes is decorated by p itself. The other one is decorated by the path p' which is obtained from p by applying a reflection through the hyperplane $\mathfrak{h}$ to all the vertices of p after the vertex $p(k_s)$. The hyperplane $\mathfrak{h}$ corresponds to the s-th hyperplane touched by p. It is shown in [BCS17, Proposition 4.11] that the paths decorating the leaves of the three are exactly those associated to standard tableaux with residue sequence $\boldsymbol{i}^{\boldsymbol{\lambda}}$. The endpoint of all of these paths belongs to $W_l \cdot \boldsymbol{\lambda}$, thus proving (1).

By (3.38) we know that

$$\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \sum_{\mathfrak{t} \in \operatorname{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})} v^{\deg(\mathfrak{t})}.$$

That the right-hand side coincides with the coefficient of $H_{w_{\boldsymbol{\mu}}}$ in the expansion of $H_{\underline{w_{\boldsymbol{\lambda}}}}$ in terms of the standard basis of the Hecke algebra of W_l follows by combining Proposition 2.14 and Proposition 4.11 in [BCS17]. Equation (4.7) is now a consequence of (2.11). □

Corollary 4.10. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P_1^l(n)$ with $\boldsymbol{\lambda}$ regular. Then,* $\mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \neq \emptyset$ *if and only if $\boldsymbol{\mu}$ is in the orbit of $\boldsymbol{\lambda}$ and $w_{\boldsymbol{\mu}} \leq w_{\boldsymbol{\lambda}}$.*

Proof: It is well-known that H_x occurs in the expansion of $H_{\underline{w}}$ if and only if $x \leq w$, for any $x, w \in W_l$ and any reduced expression $\underline{w}$ of w. The result is now a direct consequence of Theorem 4.9. □

The following result provides evidence in favor of the conjectures enunciated in the introduction.

Theorem 4.11. *Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in P_1^l(n)$. Suppose that $\boldsymbol{\lambda}$ is regular. Then,*

$$\mathbb{F}_p \otimes_R \mathrm{End}_{\mathcal{H}}(BS(\underline{w}_{\boldsymbol{\lambda}})) \cong B_{l,n}^{p,\boldsymbol{\lambda}}(\kappa), \tag{4.8}$$

as graded vector spaces.

Proof: By combining (1.5) and Theorem 4.9 we have

$$\begin{aligned} \dim_v \left(\mathbb{F}_p \otimes_R \mathrm{End}_{\mathcal{H}}(BS(\underline{w}_{\boldsymbol{\lambda}}))\right) &= \textstyle\sum_{w \leq w_{\boldsymbol{\lambda}}} \left(\deg \mathbb{L}_{\underline{w_{\boldsymbol{\lambda}}}}(w_{\boldsymbol{\mu}})\right)^2 \\ &= \textstyle\sum_{\boldsymbol{\mu} \in P(\boldsymbol{\lambda})} \left(\dim_v \Delta_{\boldsymbol{\lambda}}^p(\boldsymbol{\mu})\right)^2 \qquad (4.9) \\ &= \dim_v \left(B_{l,n}^{p,\boldsymbol{\lambda}}(\kappa)\right). \end{aligned}$$

The result follows. □

Lemma 4.12. *Let $\boldsymbol{\lambda} \in P_1^l(n)$ regular and $\boldsymbol{\mu} \in P(\boldsymbol{\lambda})$. Then, $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$.*

Proof: Suppose by contradiction that there exists $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ with $\boldsymbol{\lambda} \not\trianglelefteq \boldsymbol{\mu}$. Let A and B be the boxes occupied by n in in $\mathfrak{t}^{\boldsymbol{\lambda}}$ and $\mathfrak{t}$, respectively. Note that A is the least dominant box in $[\boldsymbol{\lambda}]$ and that $\mathrm{res}(A) = \mathrm{res}(B)$ since $\boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$. Let $\bar{\boldsymbol{\lambda}}$ be the partition obtained from $\boldsymbol{\lambda}$ by removing the box A. Similarly, let $\bar{\mathfrak{t}}$ be the tableau obtained from $\mathfrak{t}$ by removing the box B and $\bar{\boldsymbol{\mu}} = \mathrm{Shape}(\bar{\mathfrak{t}})$. Let C and D be the least dominant boxes in $[\bar{\boldsymbol{\lambda}}]$ and $[\bar{\boldsymbol{\mu}}]$, respectively. If $\bar{\boldsymbol{\lambda}} \trianglelefteq \bar{\boldsymbol{\mu}}$ then $C \trianglelefteq D$. Therefore, the addable boxes to $[\bar{\boldsymbol{\mu}}]$ are less dominant than A or are located one row below than A. In the later case these boxes cannot have the same residue as A since our multicharge κ is adjacency-free. So, relocating the boxes A and B we would obtain that $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$, contradicting our assumption. Then, we must have $\bar{\boldsymbol{\lambda}} \not\trianglelefteq \bar{\boldsymbol{\mu}}$. Note that $\boldsymbol{i}^{\bar{\mathfrak{t}}} = \boldsymbol{i}^{\bar{\boldsymbol{\lambda}}}$. So, we are in the same situation as in the beginning of the proof but with one box less. The result follows now by induction. □

Corollary 4.13. *Let $\boldsymbol{\lambda} \in P_1^l(n)$ regular. Then,*

$$P(\boldsymbol{\lambda}) = \{\boldsymbol{\mu} \in P_1^l(n) \mid \boldsymbol{\mu} \in W_l \cdot \boldsymbol{\lambda} \text{ and } w_{\boldsymbol{\mu}} \leq w_{\boldsymbol{\lambda}}\} \tag{4.10}$$

and the order inherited by $P(\boldsymbol{\lambda})$ from the dominance order on $P_1^l(n)$ is a refinement of reversed Bruhat order. Concretely, if $\boldsymbol{\mu}, \boldsymbol{\nu} \in P(\boldsymbol{\lambda})$ and $w_{\boldsymbol{\nu}} \leq w_{\boldsymbol{\mu}}$ then $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\nu}$.

Proof: Equation (4.10) is just a restatement of Corollary 4.10. Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in P(\boldsymbol{\lambda})$. Suppose that $w_{\boldsymbol{\nu}} \leq w_{\boldsymbol{\mu}}$. Then, $\boldsymbol{\nu} \in P(\boldsymbol{\mu})$ and Lemma 4.12 implies $\boldsymbol{\mu} \trianglelefteq \boldsymbol{\nu}$. □

4.5. **A process to compute graded decomposition numbers.** In this section we explain a process which calculates graded decomposition numbers for $B^p_{l,n}(\kappa)$. At this point the reader should compare this process with the one described in Section 2.10 to compute p-Kazhdan-Lusztig polynomials.

Let $\boldsymbol{\lambda} \in P^l_1(n)$ regular and $\boldsymbol{\mu} \in P(\boldsymbol{\lambda})$. By combining (3.27) and Lemma 4.12 we obtain

$$\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \sum_{\substack{\boldsymbol{\nu}\in P(\boldsymbol{\lambda}) \\ \boldsymbol{\lambda} \trianglelefteq \boldsymbol{\nu} \trianglelefteq \boldsymbol{\mu}}} d^p_{\boldsymbol{\mu},\boldsymbol{\nu}} \dim_{\mathrm{v}} L^p_{\boldsymbol{\lambda}}(\boldsymbol{\nu}). \tag{4.11}$$

Let $\boldsymbol{\nu} \in P(\boldsymbol{\lambda})$. By (4.10) we know that $w_{\boldsymbol{\nu}} \leq w_{\boldsymbol{\lambda}}$. On the other hand, if $\boldsymbol{\nu} \trianglelefteq \boldsymbol{\mu}$ but $w_{\boldsymbol{\mu}} \not\leq w_{\boldsymbol{\nu}}$ we would have $\boldsymbol{\mu} \notin P(\boldsymbol{\nu})$. Then, $e(\boldsymbol{\nu})\Delta^p(\boldsymbol{\mu}) = 0$ and therefore $d^p_{\boldsymbol{\mu},\boldsymbol{\nu}} = 0$. So that, the terms on the right-hand side of (4.11) associated to $\boldsymbol{\nu}$ such that $w_{\boldsymbol{\mu}} \not\leq w_{\boldsymbol{\nu}}$ vanish and we can rewrite such an equation as

$$\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \sum_{\substack{\boldsymbol{\nu}\in W\cdot\boldsymbol{\lambda} \\ w_{\boldsymbol{\mu}} \leq w_{\boldsymbol{\nu}} \leq w_{\boldsymbol{\lambda}}}} d^p_{\boldsymbol{\mu},\boldsymbol{\nu}} \dim_{\mathrm{v}} L^p_{\boldsymbol{\lambda}}(\boldsymbol{\nu}). \tag{4.12}$$

As $d^p_{\boldsymbol{\mu},\boldsymbol{\mu}} = \dim_v L^p_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = 1$, by expanding and rearranging the terms in (4.12), we obtain

$$\dim_{\mathrm{v}} \Delta^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) - \sum_{\substack{\boldsymbol{\nu}\in W\cdot\boldsymbol{\lambda} \\ w_{\boldsymbol{\mu}} < w_{\boldsymbol{\nu}} < w_{\boldsymbol{\lambda}}}} d^p_{\boldsymbol{\mu},\boldsymbol{\nu}} \dim_{\mathrm{v}} L^p_{\boldsymbol{\lambda}}(\boldsymbol{\nu}) = \dim_v L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) + d^p_{\boldsymbol{\mu},\boldsymbol{\lambda}}. \tag{4.13}$$

Let us suppose for a moment $p = 0$. In this case it follows by [BCS17, Theorem 3.15] and [Bow17, Corollary 8.3] that $d^p_{\boldsymbol{\mu},\boldsymbol{\lambda}} \in v\mathbb{Z}[v]$, if $\boldsymbol{\mu} \neq \boldsymbol{\lambda}$. On the other hand, by general theory of graded cellular algebras [HM10, Proposition 2.18], we know that $\dim_{\mathrm{v}} L^p_{\boldsymbol{\lambda}}(\boldsymbol{\nu})$ is invariant under the involution $v \mapsto v^{-1}$. We conclude, via the algorithm described in Section 2.10 and Theorem 4.9, that

$$d^p_{\boldsymbol{\mu},\boldsymbol{\lambda}} = h^p_{w_{\boldsymbol{\mu}},w_{\boldsymbol{\lambda}}} \qquad \text{and} \qquad \dim_{\mathrm{v}} L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \mathrm{rk}_{\mathrm{v}}\, I^p_{\underline{w_{\boldsymbol{\lambda}}}}(w_{\boldsymbol{\mu}}). \tag{4.14}$$

The equality on the left-hand side of (4.14) was proven in [BCS17, Theorem 8.5]. As far as we know, the equality on the right-hand side of (4.14) had not been observed before.

Let us return to the case $p > 0$. In this setting, the inclusion $d^p_{\boldsymbol{\mu},\boldsymbol{\lambda}} \in v\mathbb{Z}[v]$ is no longer true. So, that in order to calculate graded decomposition numbers we must compute graded dimensions of simple modules. We stress that we are in the same situation as in the p-Kazhdan-Lusztig theory. Namely, in order to calculate p-Kazhdan-Lusztig polynomials we have to compute graded ranks of intersection forms.

In the light of the above, we introduce the following conjectures.

Blob vs Soergel Conjecture. *Let $\boldsymbol{\lambda} \in P^l_1(n)$ regular. Suppose that $\boldsymbol{\mu} \in P(\boldsymbol{\lambda})$. Then,*

$$d^p_{\boldsymbol{\mu},\boldsymbol{\lambda}} = h^p_{w_{\boldsymbol{\mu}},w_{\boldsymbol{\lambda}}}.$$

Blob vs Light Leaves Conjecture. *Let $\boldsymbol{\lambda} \in P^l_1(n)$ regular. Suppose that $\boldsymbol{\mu} \in P(\boldsymbol{\lambda})$. Then,*

$$\dim_{\mathrm{v}} L^p_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) = \mathrm{rk}_{\mathrm{v}}\, I^p_{\underline{w_{\boldsymbol{\lambda}}}}(w_{\boldsymbol{\mu}}).$$

By the previous discussion and Theorem 4.9 it is clear that **Blob vs Soergel Conjecture** and **Blob vs Light leaves Conjecture** are equivalent.

5. Proof of Blob vs Soergel conjecture for $\tilde{A}_1$.

In this section we prove that graded decomposition numbers of $B^p_{2,n}(\kappa)$ coincide with p-Kazhdan-Lusztig polynomials of $\tilde{A}_1$, that is, we prove conjecture Blob vs Soergel (and therefore Conjecture Blob vs Light Leaves) for $\tilde{A}_1$. So, for the rest of this section $l = 2$, $e, n \in \mathbb{N}$ and $\kappa = (\kappa_1, \kappa_2)$ is an increasing adjacency-free multicharge.

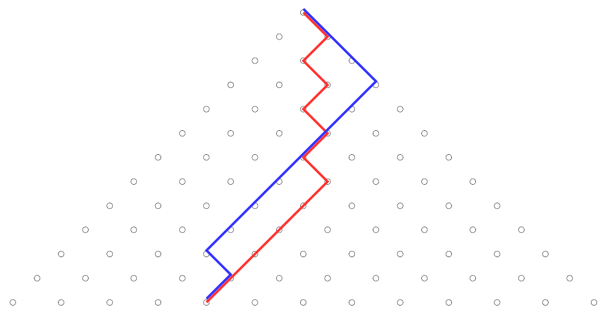
5.1. **Paths in the Pascal triangle and hooks.** In type $\tilde{A}_1$ the corresponding affine Weyl group is the infinite dihedral group $W = \langle s, t \mid s^2 = t^2 = e \rangle$. In this group, each element different from the identity has a unique reduced expression of the form

$$k_s := sts\dots\ k\,(\text{terms}) \qquad \text{and} \qquad k_t := tst\dots\ k\,(\text{terms}), \tag{5.1}$$

for some integer $k \geq 1$. We use the convention $0_s = 0_t = e$.
On the other hand, the ambient space for our alcove geometry is $\mathbb{R}^2/\langle \epsilon_1 + \epsilon_2 \rangle \simeq \mathbb{R}$. Therefore, a path $p_{\mathfrak{t}}$ associated to a standard tableau $\mathfrak{t}$ is drawn as a one-dimensional continuous path starting at $\odot$ and then going to the left or to the right according to the component occupied by the numbers in $\mathfrak{t}$. To draw paths in $\mathbb{R}$ we will represent them as paths in the Pascal triangle. Given a standard tableau $\mathfrak{t}$ we use the convention that the k-th step in the path $p_{\mathfrak{t}}$ is drawn to the right (resp. left) if k is located in the first (resp. second) component of $\mathfrak{t}$.
Each point in the Pascal triangle is determined by a *level* and a *weight*. The highest point in the Pascal triangle is at level 0. Levels increase by one from top to bottom. At level n the leftmost point has weight $-n$. Weights increase by two form left to right.
Clearly, two standard tableaux have the same shape if and only if their endpoints coincide. For this reason, we often identify this common endpoint with $\boldsymbol{\lambda}$.

Example 5.1. Let $\boldsymbol{\lambda} = (1^4, 1^8) \in P^2_1(12)$. On the left-hand side of (5.2), we have two standard tableaux $\mathfrak{s}$ and $\mathfrak{t}$ of shape $\boldsymbol{\lambda}$. On the right-hand side of (5.2), we have a Pascal triangle. For instance, the endpoint of both paths in the picture is at level 12 and weight -4.

$$\mathfrak{s} = \mathfrak{t}^{\boldsymbol{\lambda}} = \left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline 5\\\hline 7\\\hline\end{array}\ , \begin{array}{|c|}\hline 2\\\hline 4\\\hline 6\\\hline 8\\\hline 9\\\hline 10\\\hline 11\\\hline 12\\\hline\end{array} \right) \qquad \mathfrak{t} = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 11\\\hline\end{array}\ , \begin{array}{|c|}\hline 4\\\hline 5\\\hline 6\\\hline 7\\\hline 8\\\hline 9\\\hline 10\\\hline 12\\\hline\end{array} \right)$$



(5.2)

It is clear that for an arbitrary $\boldsymbol{\lambda} \in P^2_1(n)$ the path $p_{\boldsymbol{\lambda}}$ associated to the dominant tableau $\mathfrak{t}^{\boldsymbol{\lambda}}$ is the one that first zig-zags (right-left) and then finishes in a straight line ending up in $\boldsymbol{\lambda}$.

Let $\boldsymbol{\lambda} \in P^2_1(n)$ and $\mathfrak{t} \in \mathrm{Std}(\boldsymbol{\lambda})$. Suppose that k and $k+1$ are located in different components of $\mathfrak{t}$. Let $\mathfrak{s}$ be the standard tableau obtained from $\mathfrak{t}$ by interchanging k and $k+1$. In this case, we say that $\mathfrak{s}$ is obtained from $\mathfrak{t}$ by *making a hook at level k*. This name is justified by the way in which paths $p_{\mathfrak{t}}$ and $p_{\mathfrak{s}}$ are related. Indeed, if the k-th and $(k+1)$-th steps in $p_{\mathfrak{t}}$ form a subpath of the form $\langle$ then $p_{\mathfrak{s}}$ is obtained from $p_{\mathfrak{t}}$ by making a hook at level k if $p_{\mathfrak{s}}$ coincides with $p_{\mathfrak{t}}$ except in the subpath formed by the k-th and $(k+1)$-th steps where $\langle$ is replaced by $\rangle$. One could say a similar thing for a subpath of the form $\rangle$.

5.2. **An algorithm to obtain $d_{\mathfrak{t}}$.** We recall that to each standard tableau we have associated an element $d_{\mathfrak{t}} \in \mathfrak{S}_n$ defined by the rule $d_{\mathfrak{t}}\mathfrak{t}^{\boldsymbol{\lambda}} = \mathfrak{t}$. An interesting feature of the representation of $\mathfrak{t}$ as a path in the Pascal triangle is that we can obtain in a beautiful way a reduced expression for $d_{\mathfrak{t}}$ as follows:

Algorithm 5.2. In order to obtain $d_{\mathfrak{t}}$ we must follow the next steps:

- Draw the paths $p_{\mathfrak{t}}$ and $p_{\boldsymbol{\lambda}}$ associated to $\mathfrak{t}$ and $\mathfrak{t}^{\boldsymbol{\lambda}}$, respectively.
- We will define a tuple of elements of the symmetric group $(d_{\mathfrak{t}}(0), d_{\mathfrak{t}}(1), \cdots, d_{\mathfrak{t}}(j))$ and a tuple of paths in the Pascal triangle $(p_0, p_1, \cdots, p_j)$ for some j that will be defined in the next step of the algorithm. Set $d_{\mathfrak{t}}(0) = \mathrm{id} \in S_n$ and $p_0 := p_{\boldsymbol{\lambda}}$. Assume that p_{i-1} has been defined. Define p_i as the path obtained from p_{i-1} by making a hook at any level k satisfying that the area bounded by $p_{\mathfrak{t}}$ and p_i is smaller than the one bounded by $p_{\mathfrak{t}}$ and p_{i-1}. Set $d_{\mathfrak{t}}(i) = s_k d_{\mathfrak{t}}(i-1)$.
- Repeat the previous step until $p_j = p_{\mathfrak{t}}$, for some j. Then, $d_{\mathfrak{t}} = d_{\mathfrak{t}}(j)$.

The fact that this algorithm indeed provides a reduced expression for $d_{\mathfrak{t}}$ is proved in [PRH14, Section 4]. The second step of this algorithm can be performed in several ways, so the reduced expression of $d_{\mathfrak{t}}$ obtained with this process is not unique. However, in the case considered in this section, that is $l = 2$, it is easy to see that permutations $d_{\mathfrak{t}}$ are 321-avoiding and therefore the elements of our graded cellular basis of $B^p_{2,n}(\kappa)$ do not depend on the particular choice of a reduced expression.

Example 5.3. Keeping the notation of Example 5.1 we can obtain many results, for example

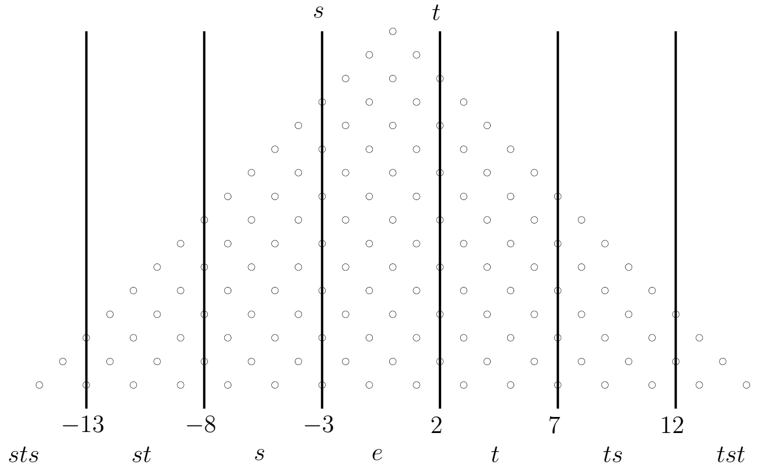
$$d_{\mathfrak{t}} = s_{10}s_9s_8s_7s_3s_4s_2 \qquad \text{or} \qquad d_{\mathfrak{t}} = s_3s_2s_4s_{10}s_9s_8s_7. \tag{5.3}$$

The reduced expression of $d_{\mathfrak{t}}$ on the left (resp. right) is obtained by applying the second step in the algorithm to the highest (resp. lowest) level possible.

5.3. **Description of standard tableaux with the same residue as $\boldsymbol{\lambda}$.** For $l = 2$ our alcove geometry lives in the real line and the set of hyperplanes reduces to the set of points $\{\mathfrak{h}^m_{1,2} := \kappa_1 - \kappa_2 + me,\ m \in \mathbb{Z}\}$. In the Pascal triangle these hyperplanes are drawn as vertical lines located on weights corresponding to the aforementioned points.

In this setting, the fundamental alcove corresponds to the one which contains the simmetry edge of the Pascal triangle. We identify affine Weyl group generators s and t with the reflections through the hyperplanes that delimit the fundamental alcove. By convention, the left hyperplane is identified with s and the right hyperplane is identified with t. Therefore, alcoves to the left (resp. right) of the fundamental alcove are labelled by elements of the form $sts\ldots$ (resp. $tst\ldots$).

Example 5.4. *Let $\kappa = (1, 4)$ and $e = 5$. In the figure there is a Pascal triangle truncated at level 15. We have drawn the six relevant hyperplanes and the labels of the alcoves appearing in the picture.*



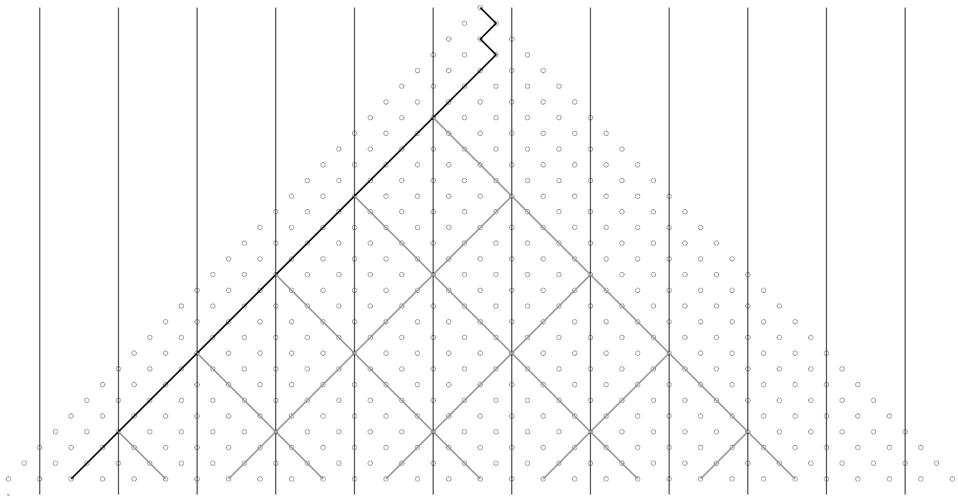
(5.4)

Given a path p we say that a subset of consecutive steps of p is a *wall to wall step* if these steps form a straight line between two consecutive hyperplanes. In the following lemma (proved in [Pla13, Lemma 4.7]), for a path $p(t)$ we think of t as a variable representing time.

Lemma 5.5. *Let $\boldsymbol{\lambda} \in P_1^2(n)$ be regular and $\mathfrak{t}$ a standard tableau with n boxes. We have that $\boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$ if and only if $p_{\mathfrak{t}}$ satisfies all of the following conditions:*

(1) *The paths $p_{\mathfrak{t}}$ and $p_{\boldsymbol{\lambda}}$ coincide until the moment of the first contact of $p_{\boldsymbol{\lambda}}$ with an hyperplane.*
(2) *Then, until the moment where $p_{\boldsymbol{\lambda}}$ touches for the last time an hyperplane, the path $p_{\mathfrak{t}}$ makes constantly wall to wall steps (as many as those made by $p_{\boldsymbol{\lambda}}$).*
(3) *After $p_{\boldsymbol{\lambda}}$ touches for the last time an hyperplane, $p_{\mathfrak{t}}$ is completed by a straight line until it stops (at level n).*

Example 5.6. Let $\kappa = (1, 4)$ and $e = 5$. In the figure it is shown a Pascal triangle truncated at level 30. Let $\boldsymbol{\lambda} = (1^2, 1^{28}) \in P_1^2(30)$. The path $p_{\boldsymbol{\lambda}}$ is drawn in black. The 2^5 descending paths following gray lines are those which correspond to standard tableaux with residue sequence equal to $\boldsymbol{i}^{\boldsymbol{\lambda}}$.



(5.5)

5.4. **Degree-zero subalgebra of $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$.** Let $\boldsymbol{\lambda} \in P_1^2(n)$. The Pascal triangle also provides us an easy way to compute the degree of the standard tableaux with residue sequence $\boldsymbol{i}^{\boldsymbol{\lambda}}$. Let $\mathfrak{t}$ be a standard tableau with n boxes such that $\boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$. We define $\delta(\mathfrak{t}) = 1$ (resp. $\delta(\mathfrak{t}) = 0$) if the straight line completing $p_{\mathfrak{t}}$ after the wall to wall steps points towards (resp. against) the symmetry edge of the Pascal triangle. We also define $w(\mathfrak{t})$ as the number of wall to wall steps in $p_{\mathfrak{t}}$ crossing through the fundamental alcove. It was proved in [Pla13, Lemma 4.9] the equality

$$\deg(\mathfrak{t}) = w(\mathfrak{t}) + \delta(\mathfrak{t}). \tag{5.6}$$

Consequently, $\deg(\mathfrak{t}) \geq 0$, for each $\mathfrak{t}$ with $\boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$. An immediate consequence of this fact is the following

Lemma 5.7. *Let $\boldsymbol{\lambda} \in P_1^2(n)$ be regular. Then the algebra $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$ is positively graded.*

By Lemma 3.7 we know that graded decomposition numbers of $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$ (and therefore those of $B_{2,n}^{p}(\kappa)$) belong to $\mathbb{N}[v]$. Therefore, in order to apply the process outlined in Section 4.5 to calculate graded decomposition numbers, we only need to know the value of $d_{\boldsymbol{\lambda},\boldsymbol{\mu}}^{p}(0)$. To do this it is enough to compute decomposition numbers of the degree-zero component of $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$, which we denote by $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$.

By Lemma 5.7, a cellular basis for $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$ is given by the elements $\psi_{\mathfrak{st}}^{\boldsymbol{\mu}}$ such that $\boldsymbol{i}^{\mathfrak{s}} = \boldsymbol{i}^{\mathfrak{t}} = \boldsymbol{i}^{\boldsymbol{\lambda}}$ and $\deg(\mathfrak{s}) = \deg(\mathfrak{t}) = 0$. In order to provide a better description of such a basis we need more notation.

Let $\boldsymbol{\lambda} \in P_1^2(n)$ regular. Given $\boldsymbol{\mu} \in P_1^2(n)$ we define

$$\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu}) = \{\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \mid \deg(\mathfrak{t}) = 0\} \tag{5.7}$$

$$P^0(\boldsymbol{\lambda}) = \{\boldsymbol{\mu} \in P_1^2(n) \mid \mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu}) \neq \emptyset\} \tag{5.8}$$

$$\mathrm{Std}_{\boldsymbol{\lambda}}^0 = \bigcup_{\boldsymbol{\mu} \in P^0(\boldsymbol{\lambda})} \mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu}). \tag{5.9}$$

We can now reformulate the preceding paragraph.

Lemma 5.8. *Let $\boldsymbol{\lambda} \in P_1^2(n)$ regular. Then,*

$$\{\psi_{\mathfrak{st}}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in P^0(\boldsymbol{\lambda}) \text{ and } \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu})\} \tag{5.10}$$

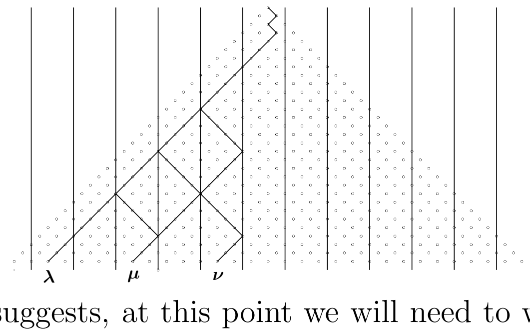
is a cellular basis for $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$.

We now compute the dimension of $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$. By (5.6) we have that $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}^0$ if and only if the path $p_{\mathfrak{t}}$ associated to $\mathfrak{t}$ satisfies the three conditions in Lemma 5.5 together the additional conditions:

(1) Wall to wall steps of $p_{\mathfrak{t}}$ do not pass through the fundamental alcove.
(2) The final straight line that completes $p_{\mathfrak{t}}$, as in Lemma 5.5 (3), moves away from the symmetry edge of the Pascal triangle.

Example 5.9. Let $\kappa = (1, 4)$ and $e = 5$. In the figure it is shown a Pascal triangle truncated at level 30. Let $\boldsymbol{\lambda} = (1^2, 1^{28}) \in P_1^2(30)$, so $w_{\boldsymbol{\lambda}} = 5_s$. We have depicted the set of all paths that are associated to a standard tableau in $\mathrm{Std}_{\boldsymbol{\lambda}}^0$. It follows that $P^0(\boldsymbol{\lambda}) = \{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}\}$, where $\boldsymbol{\mu} = (1^7, 1^{23})$ and $\boldsymbol{\nu} = (1^{12}, 1^{18})$. We have $w_{\boldsymbol{\mu}} = 3_s$, $w_{\boldsymbol{\nu}} = 1_s$, $|\,\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\lambda})| = 1$, $|\,\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu})| = 3$ and $|\,\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\nu})| = 2$.

Consider for a moment, a curious coincidence of numbers (in the next lemma we give a full explanation). There are $3 = |P^0(\boldsymbol{\lambda})|$ two-column partitions: $\lambda := (1^4)$, $\mu := (2^1, 1^2)$ and $\nu := (2^2)$. The number of standard tableaux of their shapes are: 1 of shape λ, 3 of shape μ and 2 of shape ν, which is the same as $|\,\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\lambda})|$, $|\,\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu})|$ and $|\,\mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\nu})|$.



(5.11)

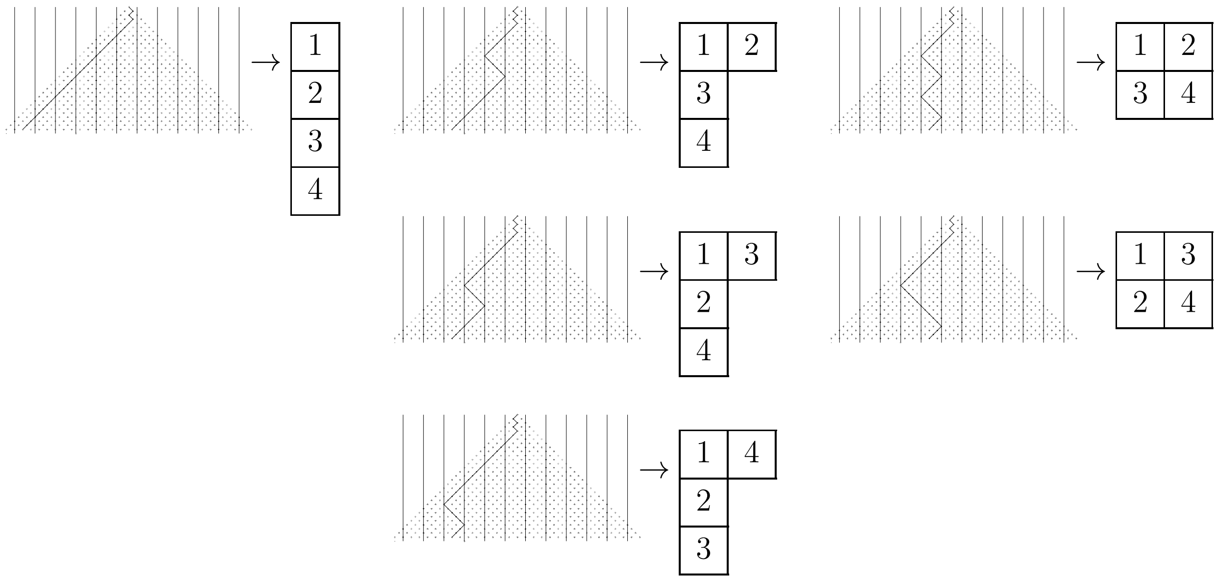
As the previous example suggests, at this point we will need to work with two-column partitions. The reader should notice the difference between such partitions and one-column bipartitions, that is, the elements in $P_1^2(n)$. Following the previous notation, we reserve bold symbols for one-column bipartitions and normal (non-bold) symbols for two-column partitions. Given a positive integer n we denote by $P_2(n)$ the set of all two-column partitions of n and by $\mathrm{Std}(\lambda)$ the set of all standard tableaux of shape λ, for $\lambda \in P_2(n)$. We also stress that an element $\lambda \in P_2(n)$ is given by $\lambda = (2^j, 1^{n-2j})$, for some $0 \leq j \leq \lceil \frac{n-1}{2} \rceil$. The following lemma is an easy consequence of the preceeding discussion.

Lemma 5.10. *Let $\boldsymbol{\lambda} \in P_1^2(n)$ regular. Assume that $w_{\boldsymbol{\lambda}} = k_s$, for some $k \geq 2$. Then,*

(1) $P^0(\boldsymbol{\lambda}) = \{\boldsymbol{\mu} \in P_1^2(n) \mid \boldsymbol{\mu} \in W \cdot \boldsymbol{\lambda}$ *and* $w_{\boldsymbol{\mu}} = (k-2j)_s$ *for some* $0 \leq j \leq \lceil \frac{k-2}{2} \rceil\}$.
(2) *There is a bijection* $P^0(\boldsymbol{\lambda}) \to P_2(k-1)$ *given by* $\boldsymbol{\mu} \to \mu$, *where* $w_{\boldsymbol{\mu}} = (k-2j)_s$ *and* $\mu = (2^j, 1^{k-1-2j})$.
(3) *Given* $\boldsymbol{\mu} \in P^0(\boldsymbol{\lambda})$ *with* $w_{\boldsymbol{\mu}} = (k-2j)_s$, *there is a bijection* $\mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \to \mathrm{Std}(\mu)$ *denoted by* $\mathfrak{t} \to \tau_{\mathfrak{t}}$, *determined by the following rule: An integer* $1 \leq j \leq k-1$ *is located in the second column of* $\tau_{\mathfrak{t}}$ *if and only if the* j*-th wall to wall step of* $p_{\mathfrak{t}}$ *points towards the symmetry edge of the Pascal triangle.*
(4) *The above statements remain true if we replace* s *by* t.

Example 5.11. Let us illustrate Lemma 5.10(3) with an example. We keep parameters and notation of Example 5.9. The bijections are given as follows:

$\mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) \to \mathrm{Std}(\lambda)$ $\qquad$ $\mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\mu}) \to \mathrm{Std}(\mu)$ $\qquad$ $\mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\nu}) \to \mathrm{Std}(\nu)$



Corollary 5.12. *Let* $\boldsymbol{\lambda} \in P_1^2(n)$ *regular with* $w_{\boldsymbol{\lambda}} = k_s$ *(or* $w_{\boldsymbol{\lambda}} = k_t$*), for some* $k \geq 2$. *Then,* $\dim B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0 = C_{k-1}$, *where* C_{k-1} *denotes the* $(k-1)$*-th Catalan number.*

Proof: The result follows by combining Lemma 5.8, Lemma 5.10 and the well-known formula

$$C_{k-1} = \sum_{j=0}^{\lceil \frac{k-2}{2} \rceil} |\,\mathrm{Std}(2^j, 1^{k-1-2j})|^2. \tag{5.12}$$

□

Remark 5.13. For $\boldsymbol{\lambda}$ regular and $w_{\boldsymbol{\lambda}} = k_s$ (or $w_{\boldsymbol{\lambda}} = k_t$), the condition $k \geq 2$ in the previous results is not restrictive since if $0 \leq k < 2$ it is easy to see that $\mathrm{Std}^0_{\boldsymbol{\lambda}} = \{\mathfrak{t}^{\boldsymbol{\lambda}}\}$, and therefore $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0 \cong \mathbb{F}_p$.

5.5. $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$ **and the Temperley-Lieb algebra.** We will now focus on to determine a presentation for $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$. We start by defining certain elements $\mathbf{U}_j^{\boldsymbol{\lambda}} \in B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$. We will show later that such elements actually generate $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$. In order to introduce such elements we first need the following

Definition 5.14. Let $\boldsymbol{\lambda} = (1^{\lambda_1}, 1^{\lambda_2}) \in P_1^2(n)$ regular with $w_{\boldsymbol{\lambda}} = k_s$ (or $w_{\boldsymbol{\lambda}} = k_t$), for some $k \geq 1$. We set $m_{\boldsymbol{\lambda}} = \min\{\lambda_1, \lambda_2\}$ and define

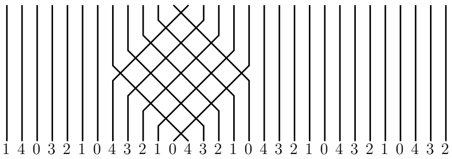
$$f_{\boldsymbol{\lambda}} = \begin{cases} 2m_{\boldsymbol{\lambda}} - (\kappa_1 - \kappa_2), & \text{if } m_{\boldsymbol{\lambda}} = \lambda^1; \\ 2m_{\boldsymbol{\lambda}} + (\kappa_1 - \kappa_2) + e, & \text{if } m_{\boldsymbol{\lambda}} = \lambda^2. \end{cases} \tag{5.13}$$

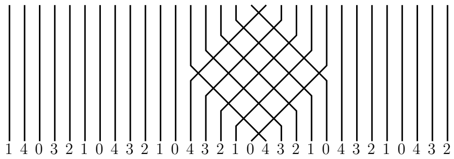
The number $f_{\boldsymbol{\lambda}}$ corresponds to the level when for first time $p_{\boldsymbol{\lambda}}$ touches a hyperplane. Given an integer $1 \leq j < k-1$ we define $\underline{j} = f_{\boldsymbol{\lambda}} + je$ and $\mathbf{U}_j^{\boldsymbol{\lambda}} \in B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$ as

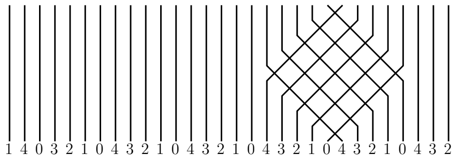
$$(\psi_{\underline{j}})(\psi_{\underline{j}-1}\psi_{\underline{j}+1})(\psi_{\underline{j}-2}\psi_{\underline{j}}\psi_{\underline{j}+2})\cdots(\psi_{\underline{j}-e+1}\psi_{\underline{j}-e+3}\cdots\psi_{\underline{j}+e-3}\psi_{\underline{j}+e-1})\cdots(\psi_{\underline{j}-2}\psi_{\underline{j}}\psi_{\underline{j}+2})(\psi_{\underline{j}-1}\psi_{\underline{j}+1})(\psi_{\underline{j}})e(\boldsymbol{\lambda}).$$

We refer to elements $\mathbf{U}_j^{\boldsymbol{\lambda}}$ as the diamond of weight $\boldsymbol{\lambda}$ at position $\underline{j}$. The following example justifies such a name, as well as the fact that $\mathbf{U}_j^{\boldsymbol{\lambda}} \in B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$.

Example 5.15. Let $\kappa = (1,4)$ and $e = 5$. Let $\boldsymbol{\lambda} = (1^2, 1^{28}) \in P_1^2(30)$. In this case we have $w_{\boldsymbol{\lambda}} = ststs = 5_s$ and $f_{\boldsymbol{\lambda}} = 7$. Then, we have three underlined integers given by $\underline{1} = 12$, $\underline{2} = 17$ and $\underline{3} = 22$. The elements $\mathbf{U}_j^{\boldsymbol{\lambda}}$ are depicted below.

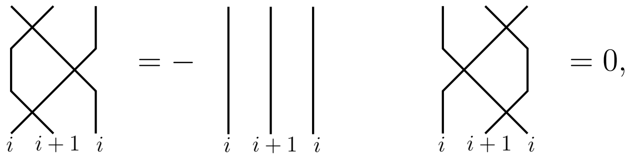
$\mathbf{U}_1^{\boldsymbol{\lambda}} =$ 

$\mathbf{U}_2^{\boldsymbol{\lambda}} =$ 

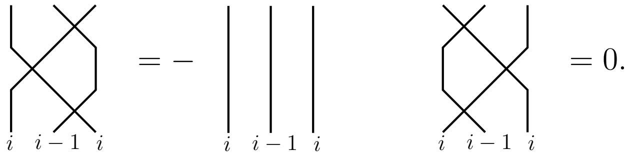
$\mathbf{U}_3^{\boldsymbol{\lambda}} =$ 

Our next goal is to show that the elements $\mathbf{U}_j^{\boldsymbol{\lambda}}$ generate $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$. We should start by proving that $\mathbf{U}_j^{\boldsymbol{\lambda}}$ actually belongs to $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$, that is, $\deg \mathbf{U}_j^{\boldsymbol{\lambda}} = 0$. However, we prefer to begin by finding out the relations satisfied by such elements. The fact that $\deg \mathbf{U}_j^{\boldsymbol{\lambda}} = 0$ will follow from such relations. Before embarking in the proof of the relations satisfied by the $\mathbf{U}_j^{\boldsymbol{\lambda}}$'s we need two lemmas.

Lemma 5.16. *In $B_{2,n}^{p}(\kappa)$ the diagrammatic relation (3.18) reduces to*



(5.14)



(5.15)

Proof: We will only prove (5.14), (5.15) is treated similarly. It follows by (3.18) that both relations in (5.14) are equivalent. We will show the equation on the right. By Proposition 3.9 we know that $e(\boldsymbol{i}) \neq 0$ if and only if $\boldsymbol{i} = \boldsymbol{i}^{\mathfrak{t}}$, for some standard tableau $\mathfrak{t}$. The relevant tableaux in $B_{2,n}^{p}(\kappa)$ have two components. Suppose that $\boldsymbol{i} = (\ldots, i, i, i+1, \ldots) \in I_e^n$. Then, $\boldsymbol{i} \neq \boldsymbol{i}^{\mathfrak{t}}$ for all standard tableau $\mathfrak{t}$, since if two i's occur consecutively in the residue sequence of a standard tableau the next value in the sequence must be $i-1$. We conclude that $e(\boldsymbol{i}) = 0$. We notice that a subsequence of type $(i, i, i+1)$ appears in the middle of the diagram on the right-hand side of (5.14). Then, such a diagram is equal to 0. □

Lemma 5.17. *In $B_{l,n}^{p}(\kappa)$ we have*

(5.16)

Proof: The result follows by a direct application of (3.16) and (3.17). □

Proposition 5.18. *Let* $\boldsymbol{\lambda} \in P_1^2(n)$ *regular with* $w_{\boldsymbol{\lambda}} = k_s$ *(or* $w_{\boldsymbol{\lambda}} = k_t$*), for some* $k \geq 2$. *The elements* $\mathbf{U}_1^{\boldsymbol{\lambda}}, \mathbf{U}_2^{\boldsymbol{\lambda}}, \dots, \mathbf{U}_{k-2}^{\boldsymbol{\lambda}}$ *satisfy*

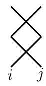
$$\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_j^{\boldsymbol{\lambda}} = \mathbf{U}_j^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}}, \qquad \text{if } |i-j|>1; \tag{5.17}$$

$$\left(\mathbf{U}_i^{\boldsymbol{\lambda}}\right)^2 = (-1)^{e-1}2\mathbf{U}_i^{\boldsymbol{\lambda}}, \tag{5.18}$$

$$\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_j^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}} = \mathbf{U}_j^{\boldsymbol{\lambda}}, \qquad \text{if } |i-j|=1; \tag{5.19}$$

Proof: Suppose that ψ_s and ψ_r occur in $\mathbf{U}_i^{\boldsymbol{\lambda}}$ and $\mathbf{U}_j^{\boldsymbol{\lambda}}$, respectively. We note that if $|i-j|>1$ then $|s-r|>1$. By (3.8) we conclude that $\mathbf{U}_i^{\boldsymbol{\lambda}}$ and $\mathbf{U}_j^{\boldsymbol{\lambda}}$ commute if $|i-j|>1$. This proves (5.17).

We will now focus on proving (5.18). We call a subdiagram of the form


(5.20)

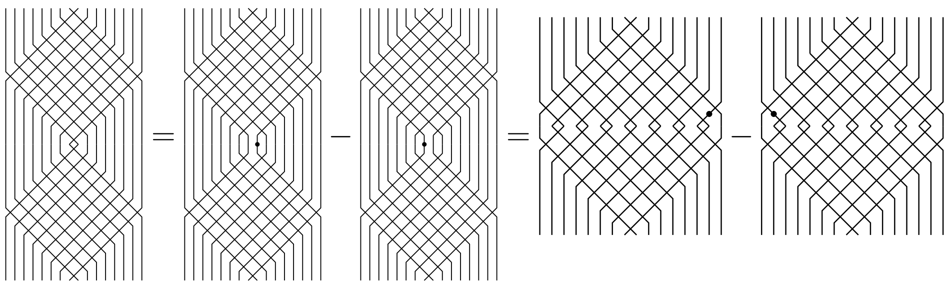
a *Double Crossing*. Such a diagram can be *replaced* by using (3.17). In accordance with such a relation we classify double crossing in four types:

(1) (ZDC) Zero Double Crossings ($i=j$);
(2) (DDC) Distant Double Crossings ($|i-j|>1$);
(3) (IADC) Increasing Adjacent Double Crossings ($j=i+1$);
(4) (DADC) Decreasing Adjacent Double Crossings ($j=i-1$).

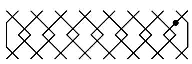
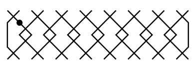
The diagram associated to $(\mathbf{U}_i^{\boldsymbol{\lambda}})^2$ is formed by two diamonds, one above the other, as is illustrated in the left-hand side of (5.21). Such a diagram assumes that $e=8$. For brevity, we have only drawn the part of the diagram where intersections occur. We have also omitted the residues on the bottom. We only need to know that they form a sequence of length $2e$ of the form

$$(a, a-1, \dots, a-(e-1), a, a-1, \dots, a-(e-1)).$$

The idea is to apply the relations to "disarm" intersections from the center to the ends. We start by applying (3.17) to the IADC just in the center of the diagram representing $(\mathbf{U}_i^{\boldsymbol{\lambda}})^2$. In this way, we obtain the second equality in (5.21). We remark that by (3.16) a dot can freely pass through an intersection as long as the residues involved in such a intersection are distinct. We use this fact $e-2$ times to move the dots up in the previous diagrams. Then, we can apply (3.17) to replace in both diagrams $(e-2)(e-1)/2-1$ DDC's by straight lines. This shows the third equation in (5.21).

$(\mathbf{U}_i^{\boldsymbol{\lambda}})^2 =$ 
(5.21)

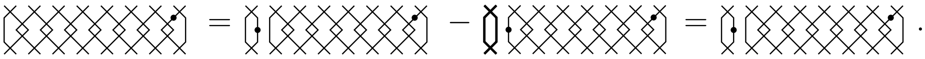
Let us now concentrate on the sub-diagrams in the middle of the rightmost diagrams in (5.21). Concretely, we are interested in

 and  . (5.22)

We notice that in both diagrams the bottom sequence is

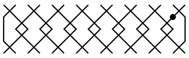
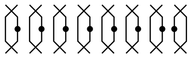
$$(a, a, a-1, a-1, \dots, a-(e-1), a-(e-1)).$$

Let us focus on the diagram on the left of (5.22). By applying (3.17) to the leftmost DADC in such a diagram we obtain the first equality of (5.23). Once again by (3.17) we obtain the second equality in (5.23), since the highlighted double-crossing is of type ZDC.

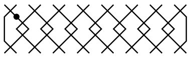
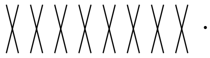


(5.23)

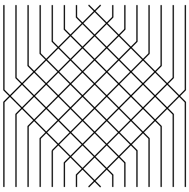
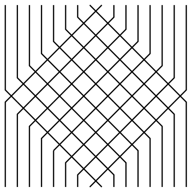
By repeating the same argument and then by applying Lemma 5.17 we obtain

 $=$  $= (-1)^{e-1}$  . (5.24)

Similar computations reveal that

 $= (-1)^{e}$  . (5.25)

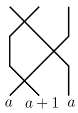
Finally, by combining (5.21), (5.24) and (5.25) we have

$(\mathbf{U}_i^{\boldsymbol{\lambda}})^2 = (-1)^{e-1}$  $- (-1)^{e}$  $= (-1)^{e-1} 2\mathbf{U}_i^{\boldsymbol{\lambda}}$. (5.26)

We now prove relation (5.19). We focus on the equation $\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_{i+1}^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}} = \mathbf{U}_i^{\boldsymbol{\lambda}}$. The relation $\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_{i-1}^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}} = \mathbf{U}_i^{\boldsymbol{\lambda}}$ is treated similarly. The diagram representing $\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_{i+1}^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}}$ is formed by three diamonds, as is illustrated in the left-hand side of (5.28). Let us refer to them as the top diamond, the middle diamond and the bottom diamond. As before, in that diagram we have only drawn the part where intersections occur and we have assumed that $e = 8$. We have also omitted the sequence of residues at the bottom of such a diagram. The only thing we need to know about such a sequence is that it is of length $3e$ and of the form

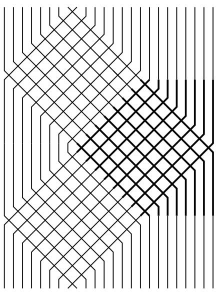
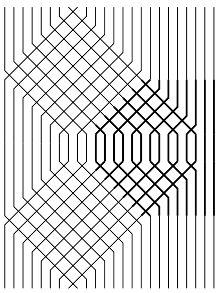
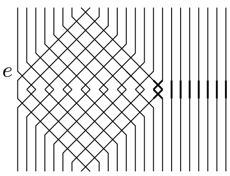
$$(a, a-1, \dots, a-(e-1), a, a-1, \dots, a-(e-1), a, a-1, \dots, a-(e-1)).$$

The intersection at the bottom corner of the top diamond, the intersection at the left corner of the middle diamond and the intersection at the top corner of the bottom diamond form a *braid* of the form

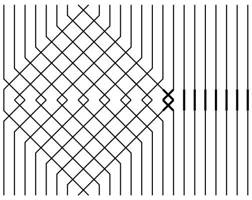
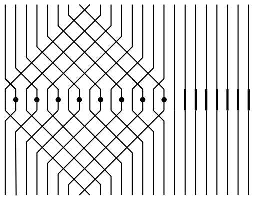
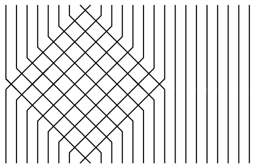


. (5.27)

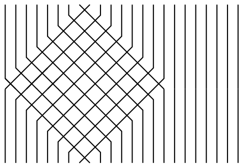
Thus, we can use (5.14) to reduce it to three straight lines multiplied by a factor (-1). Once the above braid disappeared, we have another braid of the same type located at the right of the previous one. By repeating the previous step e-times we obtain the second equality in (5.28). By applying (3.17) to the $e(e-3)$ DDC's appearing in the middle of the resultant diagram we obtain the third equality in (5.28).

$\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_{i+1}^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}} =$  $= (-1)^{e}$  $= (-1)^{e}$  . (5.28)

We now concentrate on the rightmost diagram in (5.28). The e double crossing in the middle of such a diagram are of type DADC. Once we apply (3.17) to replace them by straight lines (with some dots) we have double crossings of type ZDC. Then, only one diagram does not vanish, the one which has a dot in the middle of each double crossing of type ZDC. In this way we derive the first equality in (5.29). The second equality is a consequence of Lemma 5.17.

 $=$  $= (-1)^e$  (5.29)

Finally, by combining (5.28) and (5.29) we obtain

$\mathbf{U}_i^{\boldsymbol{\lambda}}\mathbf{U}_{i+1}^{\boldsymbol{\lambda}}\mathbf{U}_i^{\boldsymbol{\lambda}} = (-1)^{2e}$  $= \mathbf{U}_i^{\boldsymbol{\lambda}}.$ (5.30)

□

Corollary 5.19. *The elements* $\mathbf{U}_i^{\boldsymbol{\lambda}}$ *belong to* $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)_0$.

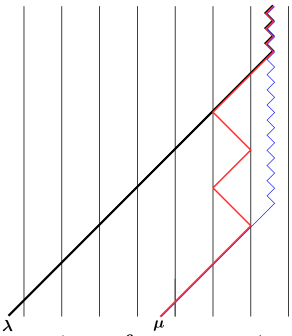
Proof: The result follows by taking the degree on both sides of (5.18). □

Given $\mu \in P_2(n)$ we define $\tau^\mu \in \mathrm{Std}(\mu)$ to be the unique standard tableau in which the numbers $1, 2, \ldots, n$ are filled in increasingly along the rows from top to bottom.

Lemma 5.20. *Let* $\boldsymbol{\lambda} \in P_1^2(n)$ *regular with* $w_{\boldsymbol{\lambda}} = k_s$ *(or* k_t*). Let* $\boldsymbol{\mu} \in P^0(\boldsymbol{\lambda})$ *with* $w_{\boldsymbol{\mu}} = (k-2j)_s$*, for some* $1 \le j \le \lceil \frac{k-2}{2} \rceil$*. Let* $\mu = (2^j, 1^{k-1-2j})$*. We denote by* $\mathfrak{t} \in \mathrm{Std}_{\boldsymbol{\lambda}}^0(\boldsymbol{\mu})$ *the tableau which is mapped to* τ^μ *via the bijection in Lemma 5.10(3). Then,*

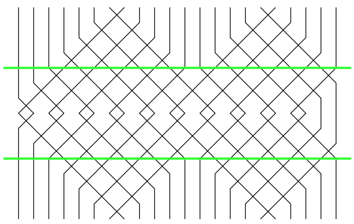
$$\psi_{\mathfrak{t}\mathfrak{t}}^{\boldsymbol{\mu}} = (-1)^{je}\mathbf{U}_1^{\boldsymbol{\lambda}}\mathbf{U}_3^{\boldsymbol{\lambda}}\cdots\mathbf{U}_{2j-1}^{\boldsymbol{\lambda}}. \tag{5.31}$$

Proof: We first notice that the restriction $1 \le j$ is made in order to discard the case $\boldsymbol{\mu} = \boldsymbol{\lambda}$. In order to know what $\psi_{\mathfrak{t}\mathfrak{t}}^{\boldsymbol{\mu}}$ is we need to know a reduced expression for $d_{\mathfrak{t}}$. We obtain such a expression by using Algorithm 5.2. To do this, we have to draw the paths $p_{\boldsymbol{\mu}}$ and $p_{\mathfrak{t}}$ associated to $\mathfrak{t}^{\boldsymbol{\mu}}$ and $\mathfrak{t}$, respectively.



(5.32)

In (5.32) we have illustrated the situation, for $w_{\boldsymbol{\lambda}} = 7_s$, $w_{\boldsymbol{\mu}} = (7 - 2 \cdot j)_s = 3_s$ and $e = 5$. So that $j = 2$. In such a figure $p_{\boldsymbol{\mu}}$ and $p_{\mathfrak{t}}$ correspond to the blue path and the red path, respectively. As a reference, we have also drawn the path $p_{\boldsymbol{\lambda}}$ associated to $\mathfrak{t}^{\boldsymbol{\lambda}}$ (the black path). We can now use Algorithm 5.2 to obtain the diagram associated to $\psi_{\mathfrak{t}\mathfrak{t}}^{\boldsymbol{\mu}}$. For instance, in the situation illustrated in (5.32) we obtain

$\psi_{\mathfrak{t}\mathfrak{t}}^{\boldsymbol{\mu}} =$  . (5.33)

Some remarks are in order. We have only drawn the part of $\psi_{\mathfrak{t}\mathfrak{t}}^{\boldsymbol{\mu}}$ where intersections occur. So that, we have both non-drawn straight lines to the left and to the right. We have also omitted the residues on the bottom of the diagram. This is not important, actually, we only need to know that the sequence below such a diagram is of the form $(a, a-1, a-2, \ldots)$. We recall the numbers in such a sequence are read modulo e. Finally, green lines do not mean anything. They are there in order to indicate a part of the diagram.

By using the same arguments as the ones utilized in the proof of Proposition 5.18 we can reduce the region delimited by the green lines to

$$(-1)^{je}$$  (5.34)

Then, (5.31) follows by combining (5.33) and (5.34). The general case follows essentially in the same way. Of course, the diagram associated to $\psi^{\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}}$ will change. For example, the height of the region that corresponds to the one bounded by the green lines depends on the distance between the symmetry edge of the Pascal triangle and the relevant wall of the fundamental alcove. Concretely, the further away the wall of the symmetry edge, the higher the region bounded by green lines will be. In any case, we still have that such a region reduces to a horizontal line of intersections, as in (5.34), proving the result. □

Corollary 5.21. *Let $\boldsymbol{\lambda} \in P^2_1(n)$ regular with $w_{\boldsymbol{\lambda}} = k_s$ (or k_t). Let $\boldsymbol{\mu} \in P^0(\boldsymbol{\lambda})$ with $w_{\boldsymbol{\mu}} = (k-2j)_s$, for some $1 \leq j \leq \lceil \frac{k-2}{2} \rceil$. Suppose that $\mathfrak{u}, \mathfrak{v} \in \mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$. Then, $\psi^{\boldsymbol{\mu}}_{\mathfrak{u}\mathfrak{v}}$ can be written as a product of elements $\mathbf{U}^{\boldsymbol{\lambda}}_j$. Consequently, the set $\{e(\boldsymbol{\lambda}), \mathbf{U}^{\boldsymbol{\lambda}}_1, \ldots, \mathbf{U}^{\boldsymbol{\lambda}}_{k-2}\}$ generates $B^{p,\boldsymbol{\lambda}}_{2,n}(\kappa)_0$.*

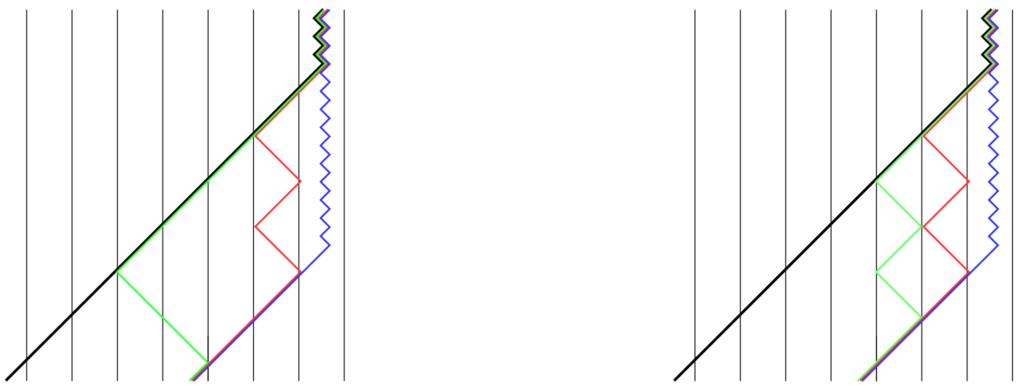
Proof: Let $\mu = (2^j, 1^{k-1-2j})$ be the two-column partition associated to $\boldsymbol{\mu}$ via Lemma 5.10(2). We also denote by $\mathfrak{t} \in \mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ the standard tableaux which is mapped to τ^{λ} via Lemma 5.10(3). A direct application of Algorithm 5.2 reveals that

$$\psi^{\boldsymbol{\mu}}_{\mathfrak{u}\mathfrak{v}} = A \psi^{\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}} B, \tag{5.35}$$

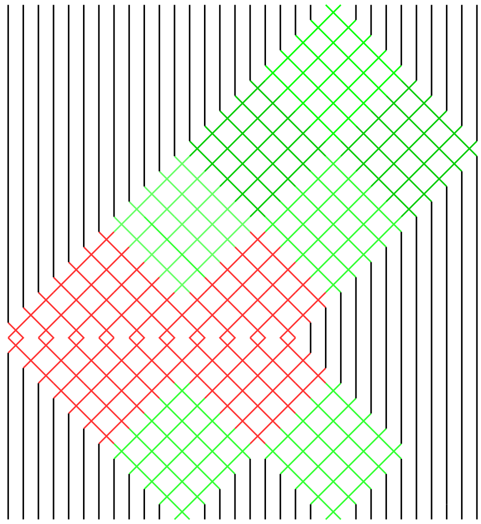
where A and B are a product of elements $\mathbf{U}^{\boldsymbol{\lambda}}_j$. Then, $\psi^{\boldsymbol{\mu}}_{\mathfrak{u}\mathfrak{v}}$ can be written as a product of elements $\mathbf{U}^{\boldsymbol{\lambda}}_j$ by Lemma 5.20.

We can rephrase the previous paragraph by saying that each one of the elements in the cellular basis of $B^{p,\boldsymbol{\lambda}}_{2,n}(\kappa)_0$ of Lemma 5.8 (with the exception of $\psi^{\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$) belongs to the subalgebra generated by the $\mathbf{U}^{\boldsymbol{\lambda}}_i$'s. The last claim in the corollary is then a consequence of the equality $e(\boldsymbol{\lambda}) = \psi^{\boldsymbol{\lambda}}_{\mathfrak{t}^{\boldsymbol{\lambda}}\mathfrak{t}^{\boldsymbol{\lambda}}}$. □

Example 5.22. Let us illustrate Corollary 5.21. We take $\kappa = (1, 4)$ and $e = 5$. Let $\boldsymbol{\lambda} = (1^3, 1^{38})$ and $\boldsymbol{\mu} = (1^{13}, 1^{18})$. We notice that $w_{\boldsymbol{\lambda}} = 7_s$ and $w_{\boldsymbol{\mu}} = 3_s$. So that, the two-column partition corresponding to $\boldsymbol{\mu}$ is $\mu = (2^2, 1^2)$. In both pictures of (5.36) we have drawn the paths associated to $\mathfrak{t}^{\boldsymbol{\lambda}}$ (black), $\mathfrak{t}^{\boldsymbol{\mu}}$ (blue) and $\mathfrak{t}$ (red), where $\mathfrak{t} \in \mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ is the tableau standard which is mapped to τ^{μ}.


(5.36)

We denote by $\mathfrak{u} \in \mathrm{Std}^0_{\boldsymbol{\lambda}}(\boldsymbol{\mu})$ (resp. $\mathfrak{v}$) the standard tableaux associated to the green path of the left (resp. right) picture. The key point here is that in both cases, the red path is in between the blue and green paths. By performing Algorithm 5.2 and ignoring the straight lines to the left and to the right in $\psi^{\boldsymbol{\mu}}_{\mathfrak{u}\mathfrak{v}}$ and also its residue sequence on the bottom, we obtain

$$\psi^{\boldsymbol{\mu}}_{\mathfrak{u}\mathfrak{v}} =$$  $$= \mathbf{U}^{\boldsymbol{\lambda}}_4\mathbf{U}^{\boldsymbol{\lambda}}_3\mathbf{U}^{\boldsymbol{\lambda}}_5\mathbf{U}^{\boldsymbol{\lambda}}_2\mathbf{U}^{\boldsymbol{\lambda}}_4\psi^{\boldsymbol{\mu}}_{\mathfrak{t}\mathfrak{t}}\mathbf{U}^{\boldsymbol{\lambda}}_2\mathbf{U}^{\boldsymbol{\lambda}}_4. \tag{5.37}$$

We are now in position to demonstrate that $B^{p,\boldsymbol{\lambda}}_{2,n}(\kappa)_0$ is isomorphic to the Temperley-Lieb algebra [TL71].

Definition 5.23. *Let n be a positive integer and $q \in \mathbb{F}_p^\times$. The Temperley-Lieb algebra $Tl^p_n(q)$ is the $\mathbb{F}_p$-algebra associative on the generators $1, U_1, \dots, U_{n-1}$ subject to the following relations*

$$U_iU_jU_i = U_i, \qquad \text{if } |i-j| = 1; \tag{5.38}$$

$$U_i^2 = -(q+q^{-1})U_i, \tag{5.39}$$

$$U_iU_j = U_jU_i, \qquad \text{if } |i-j| > 1. \tag{5.40}$$

Theorem 5.24. *Let $\boldsymbol{\lambda} \in P^2_1(n)$ regular with $w_{\boldsymbol{\lambda}} = k_s$ (or k_t), for some $k \geq 2$. We set*

$$q = \begin{cases} 1, & \text{if } e \text{ is odd;} \\ -1 & \text{if } e \text{ is even.} \end{cases}$$

Then, there exists an isomorphism of $\mathbb{F}_p$-algebras $\Phi : Tl^p_{k-1}(q) \mapsto B^{p,\boldsymbol{\lambda}}_{2,n}(\kappa)_0$ determined by $1 \mapsto e(\boldsymbol{\lambda})$ and $U_i \mapsto \mathbf{U}^{\boldsymbol{\lambda}}_i$.

Proof: By combining Proposition 5.18 and Corollary 5.21 we have that Φ is a surjective homomorphism. On the other hand, it is well-known that the dimension of $Tl^p_{k-1}(q)$ is the $(k-1)$-th Catalan number C_{k-1}. The result now follows by Corollary 5.12. □

We recall that the function $f_p : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \{0,1\}$ was defined in (2.19).

Lemma 5.25. *The Temperley-lieb algebra $Tl^p_n(\pm 1)$ is a cellular algebra with cell and simple modules parameterized by $P_2(n)$. Let $\lambda = (2^j, 1^{n-2j})$, $\mu = (2^k, 1^{n-2k}) \in P_2(n)$. The corresponding decomposition number, $d^{p,Tl}_{\lambda,\mu}$, is given by*

$$d^{p,Tl}_{\lambda,\mu} = f_p(n-2k, j-k). \tag{5.41}$$

In particular, $d^{p,Tl}_{\lambda,\mu} = 0$ or 1, for all $\lambda, \mu \in P_2(n)$.

Proof: The cellularity of $Tl^p_n(\pm 1)$ is well-known (see, for instance, [GL96]). The formula in (5.41) is implicit in [Jam87, Theorem 24.15], using the fact that $Tl^p_n(\pm 1)$ can be realized as a quotient of the group algebra $\mathbb{F}_p\mathfrak{S}_n$ (see also [CGM03, Proposition 4.5]). We remark that with respect to both references we are working in the transpose setting. □

Theorem 5.26. *Let $\boldsymbol{\lambda} \in P^2_1(n)$ regular. Suppose that $\boldsymbol{\mu}, \boldsymbol{\nu} \in P^0(\boldsymbol{\lambda})$. Let μ and ν be the two-column partitions associated to $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ according to Lemma 5.10(2). Then,*

$$d^{p,\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\nu}}(0) = d^{p,Tl}_{\mu,\nu} = h^p_{w_{\boldsymbol{\mu}},w_{\boldsymbol{\nu}}}(0). \tag{5.42}$$

Proof: The first equality follows by Theorem 5.24. The second equality follows by combining Lemma 5.25 and equation (2.20). □

Corollary 5.27. *In type $\tilde{A}_1$, Blob vs Soergel Conjecture and Blob vs Light Leaves Conjecture hold.*

Proof: On the one hand, the algorithm outlined in Section 2.10 gives us p-Kazhdan-Lusztig polynomials and graded ranks of intersection forms in type $\tilde{A}_1$. On the other hand, the algorithm outlined in Section 4.5 gives us graded decomposition numbers and graded dimensions of simple modules of $B_{2,n}^{p,\boldsymbol{\lambda}}(\kappa)$. By Theorem 4.9 and Theorem 5.26 both algorithms produce the same polynomials. □

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